SEQUENCES and SERIES.

When Carl Friederich Gauss (1777 - 1855) was just a small boy about 9 years old, he was in the class of a teacher called Mr Bruttner who gave the class an addition problem to do as a punishment. He told the class to add up all the whole numbers from 1 to 100.

Young Gauss did it in seconds, which greatly annoyed the teacher!

This was Gauss' method:

Let $S = 1 + 2 + 3 + \dots + 100$ So $S = 100 + 99 + 98 + \dots + 1$

Adding these two versions together :

 $2S = 101 + 101 + \dots + 101$ (there would be a total of 100 lots of 101)

so $2S = 101 \times 100$

and S = 5050

Similarly, the sum of numbers from 1 to 2000 is just as quick and simple!

This great idea works for all series such as the following....

Find the sum of 20 terms of this series: $S = 2 + 5 + 8 + 11 + \dots$ Firstly, we need to know what the 20th term is: Let $t_1 = 2$ $t_2 = 5 = 2 + 1 \times 3$ $t_3 = 8 = 2 + 2 \times 3$ $t_4 = 11 = 2 + 3 \times 3$ $t_5 = 14 = 2 + 4 \times 3$ so $t_{20} = 2 + 19 \times 3 = 59$ Using the same method:

 $S_{20} = 2 + 5 + 8 + \dots + 59$ $S_{20} = 59 + 56 + 2$ Adding : $2S_2 = 61 + 61 + \dots + 61 = 61 \times 20$ Divide by 2 and we get: $S_{20} = 610$

EXTRAS

A SEQUENCE is just the separate terms... 2, 5, 8, 59 A SERIES is when we ADD the terms 2 + 5 + 8 + ... + 59

The separate terms are often denoted as t_1, t_2, \dots, t_n The SUM of n terms is written as S_n

Example: Consider the sequence: 5, 9, 13,....

t ₁ = 5	
$t_2 = 9$	$= 5 + 1 \times 4$
$t_3 = 13$	$= 5 + 2 \times 4$
$t_4 = 17$	$= 5 + 3 \times 4$
$t_5 = 21$	= 5 + 4 ×4

so $t_{10} = 5 + 9 \times 4 = 41$ $t_{80} = 5 + 79 \times 4 = 321$

The nth term $t_n = 5 + (n-1) \times 4$

The SUM of 80 terms is found as follows:

 $S_{80} = 5 + \dots + 321$ $S_{80} = 321 + \dots + 1$ Adding: $2S_{80} = 321 \times 80$ So that $s_{80} = 321 \times 40 = 12840$

S₈₀ is sometimes called the *Partial Sum* of 80 terms or just *Sum* of 80 terms.

When we ADD the same number each time to find the next term, it is called an ARITHMETICAL sequence or series.

SIGMA NOTATION : (uses the Greek letter Sigma = Σ)

$$\sum_{1}^{20} (2n+1)$$

This means "find the sum of terms like 2n + 1 using values of n from 1 to 20"

If n = 1, the term is 3 If n = 2, the term is 7 If n = 20, the term is 41 The SERIES is $S_{20} = 3 + 7 + ... + 41$ so $S_{20} = 41 + + 3$ Adding we get $2S_{20} = 41 \times 20$ so that $S_{20} = 410$

GENERAL ARITHMETICAL SERIES.

Suppose the 1^{st} term is *a* and each time we add *d* to find the next term.

- $t_1 = a$ $t_2 = a + \mathbf{1} \times d$ $t_3 = a + \mathbf{2} \times d$ $t_4 = a + \mathbf{3} \times d$ $t_5 = a + \mathbf{4} \times d$
- so $t_{10} = a + \mathbf{9} \times d$ $t_{80} = a + \mathbf{79} \times d$

The n^{th} term $t_n = a + (n - 1) \times d$

The SUM of n terms is found by writing the series out backwards and adding:

 $S_n = 1^{st} term + \dots + n^{th} term = t_1 + \dots + t_n$ $S_n = n^{th} term + \dots + 1^{st} term = t_n + \dots + t_1$

Adding, we get: $2S_n = (t_1 + t_n) \times n$

So
$$S_n = \underline{n}(t_1 + t_n)$$

If we substitute $t_1 = a$ and $t_n = a + (n - 1)d$ we get:

$$S_{n} = \underline{n} \left[\begin{array}{c} a + a + (n-1)d \end{array} \right]$$
$$S_{n} = \underline{n} \left[\begin{array}{c} 2a + (n-1)d \end{array} \right]$$

This could be used to find the sum of an arithmetical series given (a) the 1^{st} term a

- (b) the number you add to get the next term *d* often called the *common difference*.
- (c) the number of terms n.

Example Find the sum of 200 terms of the series with 1^{st} term 6 and common difference of 4.

Simply substitute n = 200, a = 6, d = 4

$$S_{n} = \frac{n}{2} \left[2a + (n-1)d \right]$$
$$= \frac{200}{2} \left[12 + 199 \times 4 \right]$$

= 80800

GEOMETRIC SEQUENCES AND SERIES.

Instead of adding the same number to get the next term as in Arithmetical sequences, we MULTIPLY by the same number to get the next term. The number we multiply by is called the *common ratio*.

Suppose the 1st term is 5 and the common ratio is 3:

 $\begin{array}{l} t_{1} = 5 \\ t_{2} = 5 \times 3^{1} \\ t_{3} = 5 \times 3^{2} \\ t_{4} = 5 \times 3^{3} \\ t_{5} = 5 \times 3^{4} \\ t_{6} = 5 \times 3^{5} \end{array}$ Notice how the powers increase. $\begin{array}{l} t_{20} = 5 \times 3^{19} \\ t_{400} = 5 \times 3^{399} \\ t_{n} = 5 \times 3^{(n-1)} \end{array}$ Finding the sum of a number of terms of such a sequence can be done in a very clever way:

Consider the sum of just 4 terms of a series with 1st term of 2 and with a common ratio of 4:

S = 2 + 8 + 32 + 128 EQU 1

We MULTIPLY the series by the common ratio!

So 4S = 8 + 32 + 128 + 512 EQU 2

Now we write EQU 2 above EQU 1 positioned very cleverly as follows:

EQU 2	4S =	8 + 32 + 128 + 512
EQU 1	S = 2 - 2	+8+32+128

We now just subtract:

4S = 8 + 32 + 128 + 512 S = 2 + 8 + 32 + 128 3S = -2 + 0 + 0 + 0 + 512 3S = 510so S = 170

Obviously in this case, we could have just added the 4 numbers but this same method can be used for any number of terms!

Let's find the sum of 12 terms of the same series.

 $S = 2 + 2 \times 4^{1} + 2 \times 4^{2} + 2 \times 4^{3} + \dots 2 \times 4^{11}$ Multiply by the common ratio which is 5: $4S = 2 \times 4^{1} + 2 \times 4^{2} + 2 \times 4^{3} + \dots 2 \times 4^{11} + 2 \times 4^{12}$ Now re-write both series so that equal numbers are positioned above each other: $4S = 2 \times 4^{1} + 2 \times 4^{2} + 2 \times 4^{3} + \dots + 2 \times 4^{11} + 2 \times 4^{12}$ $S = 2 + 2 \times 4^{1} + 2 \times 4^{2} + 2 \times 4^{3} + \dots + 2 \times 4^{11}$ Subtracting : $3S = -2 + 0 + 0 + 0 + \dots + 0 + 2 \times 4^{12}$ Notice how the middle terms all disappear! $3S = 2 \times 4^{12} - 2$ 3S = 33554430 S = 11184810Notice how BIG these numbers get! These powers get so big that our calculators cannot express the sums exactly. They must be rounded and converted to *Standard Form*.

Example: Find the sum of 20 terms of the series which has 1st term of 6 and a common ratio of 5.

 $S = 6 + 6 \times 5^{1} + 6 \times 5^{2} + 6 \times 5^{3} + \dots 6 \times 5^{19}$ Multiply by the common ratio which is 5: $SS = 6 \times 5^{1} + 6 \times 5^{2} + 6 \times 5^{3} + \dots 6 \times 5^{19} + 6 \times 5^{20}$

Now re-write both series so that equal numbers are positioned above each other:

$$5S = 6 \times 5^{1} + 6 \times 5^{2} + 6 \times 5^{3} + \dots + 6 \times 5^{19} + 6 \times 5^{20}$$

$$S = 6 + 6 \times 5^{1} + 6 \times 5^{2} + 6 \times 5^{3} + \dots + 6 \times 5^{19}$$

Subtracting: $4S = -6 + 0 + 0 + 0 + ... + 0 + 6 \times 5^{20}$ $4S = 6 \times 5^{20} - 6$

If we TRY to calculate this we get...

 $4S \approx 5.722045898 \times 10^{14} - 6$ It would be pointless tying to subtract 6 from this big number! ie $4S \approx 572204589800000. - 6 !!$

The calculator can only hold the 1^{st} 10 digits. It counts the last 5 digits as zeros.

The sensible thing to do is leave the answer in factorised index form:

$$4S = 6 \times 5^{20} - 6$$

$$4S = 6(5^{20} - 6)$$

$$S = \frac{6(5^{20} - 6)}{4}$$

$$S = \frac{3(5^{20} - 6)}{2}$$

A very famous problem is this:

"I would like to offer you a choice. On 1st April, I will give you \$1, on the 2nd, I will give you \$2, on the 3rd, I will give you \$4 and carry on doubling this every day of the month. Or you can wait until the 30th of April and I will give you \$100,000! Which would you choose?" (I suspect most people would choose the \$100,000 but they would be wrong!) $S_{30} = 1 + 2 + 4 + 8 + \dots$ The series is this: $S_{30} = 1 + 2^1 + 2^2 + 2^3 + \dots + 2^{29}$ Using indices: EQU 1 $2\mathbf{S}_{30} = 2^1 + 2^2 + 2^3 + \dots + 2^{29} + 2^{30}$ EQU 2 Multiply by 2: Find EQU 2 – EQU 1 $S_{30} = 2^{30} - 1$ = 1073741824 - 1= \$1,073,741,823

Another variation of this is:

"Start a savings plan on 1st Jan for the whole 31 days. On 1st Jan you save 10c, on 2nd Jan you save 20c, on 3rd Jan you save 40c, doubling each day". If you COULD do this for 31 days, how much would you have?

The series is this:	$S_{31} = 10 + $	$-20 + 40 + 80 + \dots$			
Using indices:	$S_{31} = 10 +$	$-10 \times 2^{1} + 10 \times 2^{2} + \dots + 10 \times 2^{30}$	Equ1		
Multiply by 2:	2S ₃₁ =	$10 \times 2^{1} + 10 \times 2^{2} + \dots + 10 \times 2^{30} + 10 \times 2^{31}$	Equ 2		
Find Equ 2 – Equ 1 $S_{31} = 10 \times 2^{31} - 10$					
	$=10(2^{3})$	(1-1)			
	= 10(2)	47483647)			
	= 2147	4836470 cents			
= \$214,748,364.70 (obviously you could not keep up such a savings plan!)					

These series get to be far more interesting when the common ratio is a fraction between 1 and -1.

Consider the series $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$$= 1 + \frac{1}{2^{1}} + \frac{1}{2^{2}} + \frac{1}{2^{3}} + \frac{1}{2^{4}} + \dots$$

We will find the sum of 20 terms.

$$S_{20} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{19}}$$
 EQU 1

Multiply by the common ratio $\frac{1}{2}$

$$\frac{1}{2}S_{20} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{19}} + \frac{1}{2^{20}}$$
 EQU 2

Writing these in the same form as before but with EQU 1 on top:

$$S_{20} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{19}}$$
 EQU 1

$$\frac{1}{2}S_{20} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{19}} + \frac{1}{2^{20}}$$
 EQU 2

Subtracting : $\frac{1}{2} S_{20} = 1 + 0 + 0 + 0 + 0 + \dots + 0 - \frac{1}{2^{20}}$

$$S_{20} = 2(1 - \frac{1}{2^{20}}) \approx 1.999998093$$

This is very close to 2 because $\frac{1}{2^{20}}$ is $\frac{1}{1048576}$ which is close to zero

(In fact, it is less than *one millionth*.)

We can easily see that the sum of 30 terms would be: $S_{30} = 2(1 - \frac{1}{2^{30}}) \approx 1.99999998$ which is **even closer to 2.**

If we find the sum of 40 terms, it has too many 9's for the calculator to hold and so the calculator rounds the answer to 2!

Imagine the same series carrying on for ever with no last term:

$$S_{\infty} = 1 + \frac{1}{2^{1}} + \frac{1}{2^{2}} + \frac{1}{2^{3}} + \frac{1}{2^{4}} + \dots$$
 EQU 1

$$\frac{1}{2} \mathbf{S}_{\infty} = \frac{1}{2^{1}} + \frac{1}{2^{2}} + \frac{1}{2^{3}} + \frac{1}{2^{4}} + \dots$$
 EQU 2

Subtracting : $\frac{1}{2} S_{\infty} = 1 + 0 + 0 + 0 + 0 + 0$ for ever!

 $S_{\infty} = 2$

This is called the "*sum to infinity*". It actually means that if we keep adding on more and more terms, the sum gets closer and closer to the number 2.

We also say "the sum *approaches* 2" or "the *limit* of the sum is 2"

A general formula for the sum of a geometric series is not difficult to prove.

Consider *n* terms of
$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$
 EQU1

Multiply by r: $rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \quad EQU2$

Subtracting: $S_n - rS_n = a + \theta + \theta + \theta + \dots + \theta - ar^n$

Factorising : $S_n(1-r) = a(1-r^n)$

$$S_n = \frac{a(1-r^n)}{(1-r)} \qquad EQUA$$

N.B. This form of the equation is better if 0 < r < 1 otherwise the other version obtained by subtracting EQU1 from EQU2

ie
$$S_n = \underline{a(r^n-1)}$$
 EQUB
 $(r-1)$

EXAMPLES: *If a = 3, n = 10 and r = 5 use EQU B*

$$S_n = \frac{a(r^n - 1)}{(r - 1)} = \frac{3(5^{10} - 1)}{4} = 7324218$$

If
$$a = 3$$
, $n = 10$ and $r = \frac{1}{2}$ use EQUA
 $S_n = \frac{a(1 - r^n)}{(1 - r)} = 3\frac{(1 - \frac{1}{2}^{10})}{\frac{1}{2}} = 5.994140625$

A particularly nice proof is for the "sum to infinity".

$$S_n = a + ar + ar^2 + ar^3 + \dots EQU1$$

$$rS_n = ar + ar^2 + ar^3 + \dots EQU2$$

subtracting $S_n - rS_n = a + 0 + 0 + 0$

$$S_n(1-r) = a$$

$$S_n = \underline{a}$$
 but the series only approaches a limit if $-1 < r < 1$
 $(1-r)$

EXAMPLES

Find what limit these series would approach. 1. $S = 16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \dots$

$$S_n = \frac{a}{(1-r)} = \frac{16}{\frac{1}{2}} = 32$$

2.
$$S = 16 + 4 + 1 + \frac{1}{4} + \dots$$

$$S_n = \frac{a}{(1-r)} = \frac{16}{\frac{3}{4}} = 21\frac{1}{3}$$

3. $S = 4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \dots$

$$S_n = \frac{a}{(1-r)} = \frac{4}{1+\frac{1}{2}} = \frac{4}{1} \times \frac{2}{3} = \frac{8}{3}$$

4. $S = 4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$ note the common ratio is $\frac{3}{4}$

$$S_n = \frac{a}{(1-r)} = \frac{4}{1-\frac{3}{4}} = \frac{4}{1} \times \frac{4}{1} = 16$$

A very interesting form of infinite series is **RECURRING DECIMALS**.

$$\begin{split} S &= 0.3 + 0.03 + 0.003 + 0.0003 + \dots \\ &= 0.3 + 0.3 \times \underline{1}_{10^1} + 0.3 \times \underline{1}_{10^2} + 0.3 \times \underline{1}_{10^3} \dots \\ \end{split}$$

but there is a very nice little technique to find the sum as follows:

S = 0.33333333333333333..... EOU 1 Multiply this by 10 S = 0.33333333333333333..... EOU 1 Write EQU 1 again 9S = 3Subtract and we get: $S = \frac{1}{2}$ of course! Subtract 9S = 7So $S = \frac{7}{2}$ 3. Consider S = 0.121212121212121212...So 100S = 12.121212121212121212.... Subtract 99S = 12So S = $\frac{12}{99} = \frac{4}{33}$ 4. Consider S = 0.123123123123123123123123123... $S_0 = 123.123123123123123123123123123123123...$ Subtract 999S = 123 So S = $\frac{123}{999} = \frac{41}{333}$ Consider S = 0.142857142857142857142857...5. So 1000000S = 142857.142857142857142857142857..... Subtract 999999S = 142857 So S = $\frac{142857}{999999} = \frac{1}{7}$

Students would probably be eager to make up their own.

Subtracting
$$9S = 9$$

So $S = 1$

Another way to think of this is:

SUMMARY.

1. Find $\sum_{1}^{20}(2+3n) = 5+8+11+\ldots+62$ $S = 5 + \dots + 62$ $S = 62 + \dots + 5$ Adding $2S = 67 \times 20$ S = 6702. $\sum_{1}^{20} (2 \times 3^n) = 6 + 18 + 54 + \dots + 2 \times 3^{20}$ $S = 6 + 18 + 54 + \dots + 2 \times 3^{20}$ 3S = 18 + 54 + \dots + 2 \times 3^{20} + 2 \times 3^{21} Subtracting $2S = 2 \times 3^{21} - 6$ $2S = 2(3^{21} - 3)$ $S = (3^{21} - 3)$ 3. $\sum_{1}^{40} (3 \times 2^{n-1}) = 3 + 6 + 12 + \dots + 3 \times 2^{39}$ $S = 3 + 6 + 12 + \dots + 3 \times 2^{39}$ 2S = 6 + 12 + \dots + 3 \times 2^{39} + 3 \times 2^{40} Subtracting $S = 3 \times 2^{40} - 3$ or $3(2^{40} - 1)$ 4. $\sum_{1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$

We could just use $S_{\infty} = \frac{a}{1-r}$

$$= \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \times \frac{3}{2} = \frac{1}{2}$$

5. Find the fraction equal to 0.57575757575757.....

S = 0.5757575757575757.... 100S = 57.575757575757575757...So 99S = 57

$$S = 57 = 19$$

99 33

SUMMARY.

1. Find $\sum_{1}^{20}(2+3n) = 5+8+11+\ldots+62$

2.
$$\sum_{1}^{20} (2 \times 3^n) = 6 + 18 + 54 + \dots + 2 \times 3^{20}$$

3.
$$\sum_{1}^{40} (3 \times 2^{n-1}) = 3 + 6 + 12 + \dots + 3 \times 2^{39}$$

4.
$$\sum_{1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$$

We could just use $S_{\infty} = \frac{a}{1-r}$

5. Find the fraction equal to 0.57575757575757......