**SEQUENCES and SERIES.**

When Carl Friederich Gauss (1777 – 1855) was just a small boy about 9 years old, he was in the class of a teacher called Mr Bruttner who gave the class an addition problem to do as a punishment. He told the class to add up all the whole numbers from 1 to 100.

Young Gauss did it in seconds, which greatly annoyed the teacher!

This was Gauss’ method:

Let S = 1 + 2 + 3 + ………. + 100

So S = 100 + 99 + 98 + ………. + 1

Adding these two versions together :

2S = 101 + 101 + ……..+ 101 (there would be a total of 100 lots of 101)

so 2S = 101 × 100

and S = 5050

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Similarly, the sum of numbers from 1 to 2000 is just as quick and simple!

S = 1 + 2 + ……………. + 2000

S = 2000 + 1999 + ……………. + 1

Adding we get 2S = 2001 × 2000

so that S = 2001000

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This great idea works for all series such as the following….

Find the sum of 20 terms of this series:

S = 2 + 5 + 8 + 11 + ……

Firstly, we need to know what the 20th term is:

Let t1 = 2

t2 = 5 = 2 + **1**×3

t3 = 8 = 2 + **2**×3

t4 = 11 = 2 + **3**×3

t5 = 14 = 2 + **4**×3

so t20 = 2 + **19×**3 = 59

Using the same method:

S20 = 2 + 5 + 8 + …….. + 59

S20 = 59 + 56 + + 2

Adding : 2S2= 61 + 61 + … + 61 = 61 × 20

Divide by 2 and we get:

S20 = 610

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EXTRAS

A SEQUENCE is just the separate terms… 2, 5, 8 , …… 59

A SERIES is when we ADD the terms ….. 2 + 5 + 8 + … + 59

The separate terms are often denoted as t1, t2, …. tn

The SUM of n terms is written as Sn

Example:

Consider the sequence: 5, 9, 13,…..

t1 = 5

t2 = 9 = 5 + **1**×4

t3 = 13 = 5 + **2**×4

t4 = 17 = 5 + **3**×4

t5 = 21 = 5 + **4**×4

so t10 = 5 + **9**×4 = 41

t80 = 5 + **79**×4 = 321

The nth term tn = 5 + **(n – 1)**×4

The SUM of 80 terms is found as follows:

S80 = 5 + ……. + 321

S80 = 321 + …….. + 1

Adding: 2S80 = 321 × 80

So that s80 = 321 × 40 = 12840

S80 is sometimes called the ***Partial Sum*** of 80 terms or just ***Sum*** of 80 terms.

When we ADD the same number each time to find the next term, it is called an ARITHMETICAL sequence or series.

**SIGMA NOTATION : (uses the Greek letter Sigma = Σ)**

This means “find the sum of terms like 2n + 1 using values of n from 1 to 20”

If n = 1, the term is 3

If n = 2, the term is 7

If n = 20, the term is 41

The SERIES is S20 = 3 + 7 + … + 41

so S2­0 = 41 + ……. + 3

Adding we get 2S20 = 41×20

so that S20 = 410

**GENERAL ARITHMETICAL SERIES.**

Suppose the 1st term is ***a*** and each time we add ***d*** to find the next term.

*t1 = a*

*t2 = a +* ***1****×d*

*t3 = a +* ***2****×d*

*t4 = a +* ***3****×d*

*t5 = a +* ***4****×d*

*so t10 = a +* ***9****×d*

*t80 = a +* ***79****×d*

*The nth term tn = a +* ***(n – 1)****×d*

The SUM of n terms is found by writing the series out backwards and adding:

Sn = 1st term + ……. + nth term = ***t1 + …….. + tn***

Sn = nth term + ……. + 1st term = ***tn + …….. + t1***

Adding, we get: 2Sn = ( ***t1 + tn*** ) **× *n***

So Sn = ***n (t1 + tn)***

***2***

If we substitute t1 = ***a*** and tn = ***a + (n – 1)d***

we get:

Sn = ***n a* + *a + (n – 1)d***

***2***

Sn = ***n 2a + (n – 1)d***

***2***

This could be used to find the sum of an arithmetical series given

(a) the 1st term ***a***

(b) the number you add to get the next term ***d*** often called the ***common***

***difference.***

(c) the number of terms ***n***.

Example Find the sum of 200 terms of the series with 1st term 6 and common difference of 4.

Simply substitute ***n = 200, a = 6, d = 4***

Sn = ***n 2a + (n – 1) d***

***2***

= 200 12 + 199×4

2

= 80800

**GEOMETRIC SEQUENCES AND SERIES.**

Instead of adding the same number to get the next term as in Arithmetical sequences, we MULTIPLY by the same number to get the next term.

The number we multiply by is called the ***common ratio.***

Suppose the 1st term is 5 and the common ratio is 3:

t1 = 5

Notice how the powers increase.

t20 = 5×319

t400 = 5×3399

tn = 5×3(n – 1)

t2 = 5×31

t3 = 5×32

t4 = 5×33

t5 = 5×34

t6 = 5×35

Finding the sum of a number of terms of such a sequence can be done in a very clever way:

Consider the sum of just 4 terms of a series with 1st term of 2 and with a common ratio of 4:

S = 2 + 8 + 32 + 128 EQU 1

We MULTIPLY the series by the ***common ratio***!

So 4S = 8 + 32 + 128 + 512 EQU 2

Now we write EQU 2 above EQU 1 positioned very cleverly as follows:

EQU 2 4S = 8 + 32 + 128 + 512

EQU 1 S = 2 + 8 + 32 + 128

We now just subtract:

4S = 8 + 32 + 128 + 512

S = 2 + 8 + 32 + 128

3S = – 2 + 0 + 0 + 0 + 512

3S = 510

so S = 170

Obviously in this case, we could have just added the 4 numbers but this same method can be used for any number of terms!

Let’s find the sum of 12 terms of the same series.

S = 2 + 2×41 + 2×42 + 2×43 + ….. 2×411

Multiply by the common ratio which is 5:

4S = 2×41 + 2×42 + 2×43 + ….. 2×411 + 2×412

Now re-write both series so that equal numbers are positioned above each other:

4S = 2×41 + 2×42 + 2×43 + ….. + 2×411 + 2×412

S = 2 + 2×41 + 2×42 + 2×43 + ….. + 2×411

Subtracting : 3S = –2 + 0 + 0 + 0 + …...+ 0 + 2×412

Notice how the middle terms all disappear!

3S = 2×412 – 2

3S = 33554430

S = 11184810

Notice how BIG these numbers get!

These powers get so big that our calculators cannot express the sums exactly. They must be rounded and converted to ***Standard Form***.

Example: Find the sum of 20 terms of the series which has 1st term of 6 and a common ratio of 5.

S = 6 + 6×51 + 6×52 + 6×53 + ….. 6×519

Multiply by the common ratio which is 5:

5S = 6×51 + 6×52 + 6×53 + ….. 6×519 + 6×520

Now re-write both series so that equal numbers are positioned above each other:

5S = 6×51 + 6×52 + 6×53 + ….. + 6×519 + 6×520

S = 6 + 6×51 + 6×52 + 6×53 + ….. + 6×519

Subtracting : 4S = –6 + 0 + 0 + 0 + …. + 0 + 6×520

4S = 6×520 – 6

If we TRY to calculate this we get…

4S ≈ 5.722045898×1014 – 6

It would be pointless tying to subtract 6 from this big number!

ie 4S ≈ **572204589800000**. – 6 !!

The calculator can only hold the 1st 10 digits. It counts the last 5 digits as zeros.

The sensible thing to do is leave the answer in factorised index form:

4S = 6×520 – 6

4S = 6( 520 – 6)

S = 6( 520 – 6)

4

S = 3( 520 – 6)

2

**A very famous problem is this:**

“I would like to offer you a choice.

On 1st April, I will give you $1, on the 2nd , I will give you $2, on the 3rd, I will give you $4 and carry on doubling this every day of the month.

Or you can wait until the 30th of April and I will give you $100,000!

Which would you choose?”

(***I suspect most people would choose the $100,000 but they would be wrong***!)

The series is this: S30 = 1 + 2 + 4 + 8 + ….

Using indices: S30 = 1 + 21 + 22 + 23 + …. + 229 EQU 1

Multiply by 2: 2S30 = 21 + 22 + 23 + …. + 229 + 230 EQU 2

Find EQU 2 – EQU 1 S30 = 230 – 1

= 1073741824 – 1

= $1,073,741,823

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**Another variation of this is:**

“Start a savings plan on 1st Jan for the whole 31 days. On 1st Jan you save 10c, on 2nd Jan you save 20c, on 3rd Jan you save 40c, doubling each day”.

If you COULD do this for 31 days, how much would you have?

The series is this: S31 = 10 + 20 + 40 + 80 + ….

Using indices: S31 = 10 + 10×21 + 10×22 + …. + 10×230 Equ1

Multiply by 2: 2S31 = 10×21 + 10×22 + …. + 10×230 +10×231 Equ 2

Find Equ 2 – Equ 1 S31 = 10×231 – 10

= 10(231 – 1)

= 10(2147483647)

= 21474836470 cents

= $214,748,364.70

(***obviously you could not keep up such a savings plan***!)

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**These series get to be far more interesting when the common ratio is a fraction between 1 and –1.**

Consider the series S = 1 + 1 + 1 + 1 + 1 + …..

2 4 8 16

= 1 + 1 + 1 + 1 + 1 + …..

21 22 23 24

We will find the sum of 20 terms.

S20 = 1 + 1 + 1 + 1 + 1 + ….. 1 EQU 1

21 22 23 24  219

Multiply by the common ratio ½

½ S20 = 1 + 1 + 1 + 1 + ….. 1 + 1 EQU 2

21 22 23 24  219 220

Writing these in the same form as before but with EQU 1 on top:

S20 = 1 + 1 + 1 + 1 + 1 + ….. 1 EQU 1

21 22 23 24  219

½ S20 = 1 + 1 + 1 + 1 + ….. 1 + 1 EQU 2

21 22 23 24  219 220

Subtracting : ½ S20 = 1 + 0 + 0 + 0 + 0 + …. + 0 – 1

220

S20  = 2( 1 – 1 ) ≈ 1.999998093

220

This is **very close to 2** because 1 is 1 which is close to zero

220  1048576

(In fact, it is less than ***one millionth*.**)

We can easily see that the sum of 30 terms would be:

S30= 2( 1 – 1 ) ≈ 1.999999998 which is **even closer to 2.**

2**30**

If we find the sum of 40 terms, it has too many 9’s for the calculator to hold and so the calculator rounds the answer to 2!

**Imagine the same series carrying on for ever with no last term:**

S∞ = 1 + 1 + 1 + 1 + 1 + ….. EQU 1

21 22 23 24

½ S∞ = 1 + 1 + 1 + 1 + …… EQU 2

21 22 23 24

Subtracting : ½ S∞ = 1 + 0 + 0 + 0 + 0 + 0……for ever!

S∞ = 2

This is called the “***sum to infinity***”.

It actually means that if we keep adding on more and more terms, the sum gets closer and closer to the number 2.

We also say “the sum ***approaches*** 2” or “the ***limit*** of the sum is 2”

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A general formula for the sum of a geometric series is not difficult to prove.

Consider ***n*** terms of ***Sn***  = ***a + ar + ar2 + ar3 + … + arn – 1  EQU 1***

Multiply by ***r :***  ***rSn*** = ***ar + ar2 + ar3 + … + arn – 1  + arn EQU 2***

Subtracting : ***Sn - rSn*** = ***a + 0 + 0 + 0 + … + 0 – arn***

Factorising : ***Sn(1 – r) = a( 1 – rn )***

***Sn = a ( 1 – rn)*** ***EQU A***

***(1 – r)***

***N.B. This form of the equation is better if 0 < r < 1 otherwise the other version obtained by subtracting EQU1 from EQU2***

***ie Sn = a ( rn – 1)*** ***EQU B***

***(r – 1)***

EXAMPLES:

***If a = 3, n = 10 and r = 5 use EQU B***

***Sn = a ( rn – 1)*** = ***3(510 – 1) = 7324218***

***(r – 1) 4***

***If a = 3, n = 10 and r = ½ use EQU A***

***Sn = a ( 1 – rn)*** = ***3(1 – ½10)*** ***= 5.994140625***

***(1 – r) ½***

A particularly nice proof is for the “sum to infinity”.

***Sn***  = ***a + ar + ar2 + ar3 + … EQU 1***

***rSn*** = ***ar + ar2 + ar3 + …EQU 2***

***subtracting Sn - rSn*** = ***a + 0 + 0 + 0***

***Sn(1 – r) = a***

***Sn = a but the series only approaches a limit if -1 < r < 1***

***(1 – r)***

***EXAMPLES***

***Find what limit these series would approach.***

***1. S = 16 + 8 + 4 + 2 + 1 + ½ + ……***

***Sn = a = 16 = 32***

***(1 – r) ½***

***2. S = 16 + 4 + 1 + ¼ + ….***

***Sn = a = 16 = 21⅓***

***(1 – r) ¾***

***3. S = 4 – 2 + 1 – ½ + ¼ - ….***

***Sn = a = 4 = 4 × 2 = 8***

***(1 – r) 1 + ½ 1 3 3***

***4. S =* 4 + 3 *+ +* + …… note the common ratio is**

***Sn = a = 4 = 4 × 4 = 16***

***(1 – r) 1 – ¾ 1 1***

**A very interesting form of infinite series is RECURRING DECIMALS.**

1. Consider S = 0.333333333333333…..

Obviously we could go to the trouble of writing this as an actual series:

S = 0.3 + 0.03 + 0.003 + 0.0003 + …..

= 0.3 + 0.3×1 + 0.3×1 + 0.3×1 …….

101 102 103

but there is a very nice little technique to find the sum as follows:

S = 0.33333333333333333…… EQU 1

Multiply this by 10

So 10S = 3.33333333333333333….. EQU 2

Write EQU 1 again S = 0.33333333333333333…… EQU 1

Subtract and we get: 9S = 3

S = of course!

2. Consider S = 0.77777777777777777777777…..

So 10S = 7.777777777777777777777777…..

Subtract 9S = 7

So S =

3. Consider S = 0.121212121212121212…….

So 100S = 12.121212121212121212…..

Subtract 99S = 12

So S = =

4. Consider S = 0.123123123123123123123123123…….

So 1000S = 123.123123123123123123123123123…..

Subtract 999S = 123

So S = =

5. Consider S = 0.142857142857142857142857……

So 1000000S = 142857.142857142857142857142857……

Subtract 999999S = 142857

So S = =

Students would probably be eager to make up their own.

6. A VERY interesting case is S = 0.9999999999999999999….

Multiplying by 10, we get 10S = 9.9999999999999999999….

Subtracting 9S = 9

So S = 1

People naturally think that 0.9999999999999999999…. must be slightly less than 1.

Another way to think of this is:

If ⅓ = 0.3333333333333333333…. for ever

Then 3 ×⅓ = 0.999999999999999999999…. for ever

But 3 × ⅓ = 1 = 0.99999999999999999999….. !!!

**SUMMARY.**

1. Find = 5 + 8 + 11 +… + 62

S = 5 + ……. + 62

S = 62 + ……. + 5

Adding 2S = 67 × 20

S = 670

2. = 6 + 18 + 54 + ….. + 2×320

S = 6 + 18 + 54 + ….. + 2×320

3S = 18 + 54 + ….. + 2×320 + 2×321

Subtracting 2S = 2×321 – 6

2S = 2(321 – 3)

S = (321 – 3)

3. = 3 + 6 + 12 + …… + 3×239

S = 3 + 6 + 12 + …… + 3×239

2S = 6 + 12 + …… + 3×239 + 3×240

Subtracting S = 3×240 – 3 or 3(240 – 1)

4. = + + + + …….

We could just use S∞ = ***a***

***1 – r***

= = =

5. Find the fraction equal to 0.575757575757……..

S = 0.57575757575757…..

100S = 57.5757575757575757…

So 99S = 57

S = 57 = 19

99 33

**SUMMARY.**

1. Find = 5 + 8 + 11 +… + 62

2. = 6 + 18 + 54 + ….. + 2×320

3. = 3 + 6 + 12 + …… + 3×239

4. = + + + + …….

We could just use S∞ = ***a***

***1 – r***

5. Find the fraction equal to 0.575757575757……..