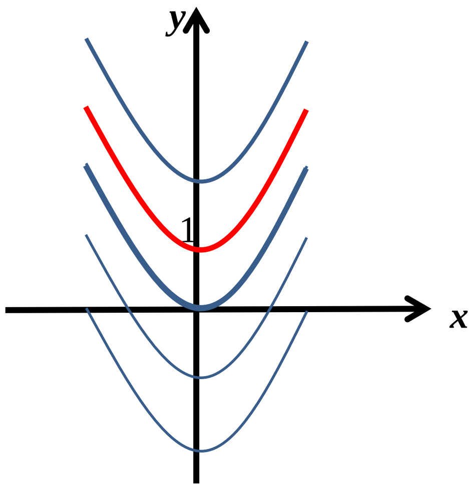
DIFFERENTIAL EQUATIONS and GRAPHS of DERIVATIVES.

1. Consider the differential equation $\frac{dy}{dx} = 2x$

If we antidifferentiate (or Integrate), then the equation for y could be ….

$y = x^2$ **or** $y = x^2 + 1$ **or** $y = x^2 + 2$ **or** $y = x^2 - 99$ **or** $y = x^2 \pm$ ***any number!***

This means a whole **family of curves** have the same derivative $\frac{dy}{dx} = 2x$



So the antiderivative of $\frac{dy}{dx} = 2x$

is $y = x^2 + c$ where "c" is any number.

This is called the GENERAL SOLUTION of the differential equation.

However, if we have an additional condition such as $y = 5$ when $x = 2$ then we can find what the "c" is equal to.

Subs $y = 5, x = 2$ *in* $y = x^2 + c$

$5 = 4 + c$ ***so*** $c = 1.$

We now say that the PARTICULAR SOLUTION is $y = x^2 + 1$

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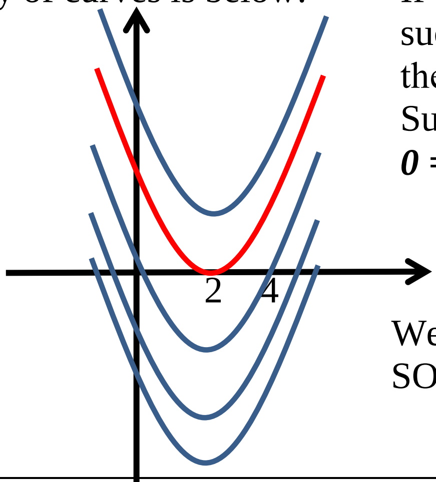
2. Similarly, if $\frac{dy}{dx} = 2x - 4$ then the general solution is $y = x^2 - 4x + c$

The family of curves is below:



If we have an additional condition such as $y = 0$ when $x = 2$ then we can find what the "c" is equal to.

Subs $y = 0, x = 2$ *in* $y = x^2 - 4x + c$

$0 = 4 - 8 + c$ ***so*** $c = 4.$

We now say that the PARTICULAR SOLUTION is $y = x^2 - 4x + 4$

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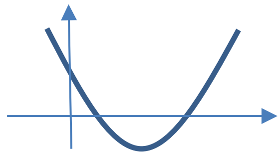
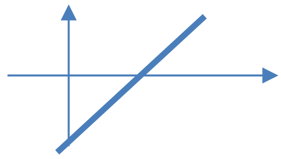
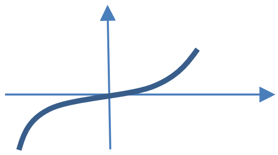
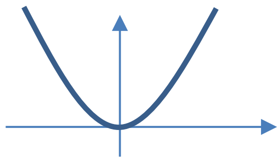
3. If $\frac{dy}{dx} = 3x^2$ find the **general solution** for y.

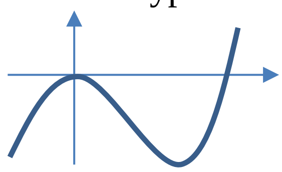
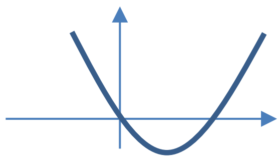
Sketch the "family" of curves.

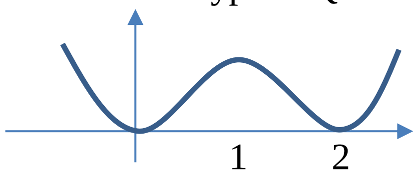
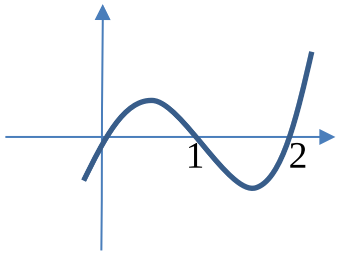
Find the **particular solution** for when $x = 2, y = 13$

4. Notice what happens to a polynomial when we **DIFFERENTIATE** it:

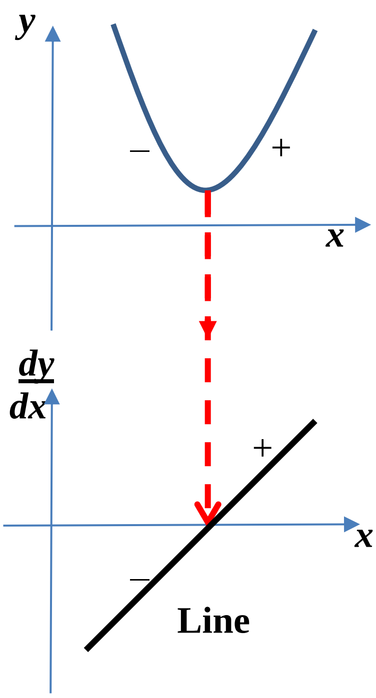
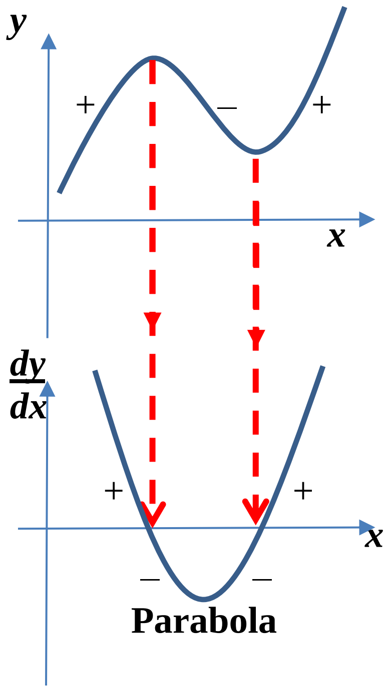
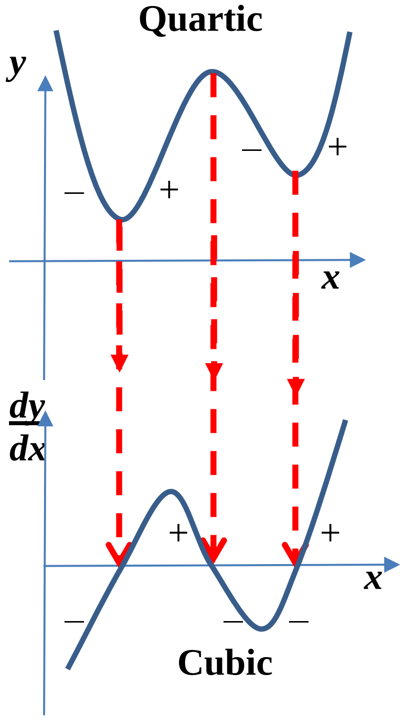
<table>
<tr><td>Quadratic Equation
$y = x^2 - 4x + 3 = (x - 1)(x - 3)$</td><td>Type of curve : Parabola
</td></tr>
<tr><td>Derivative
$\frac{dy}{dx} = 2x - 4$</td><td>Type of curve : Line
</td></tr>
<tr><td colspan="2">So if we differentiate a Quadratic Equation it becomes a Line Equation.</td></tr>
</table>

<table>
<tr><td>Basic Cubic Equation
$y = x^3$</td><td>Type of curve : Basic CUBIC
</td></tr>
<tr><td>Derivative
$\frac{dy}{dx} = 3x^2$</td><td>Type of curve : Parabola (quadratic)
</td></tr>
<tr><td colspan="2">So if we differentiate a CUBIC equation it becomes a QUADRATIC equation.</td></tr>
</table>

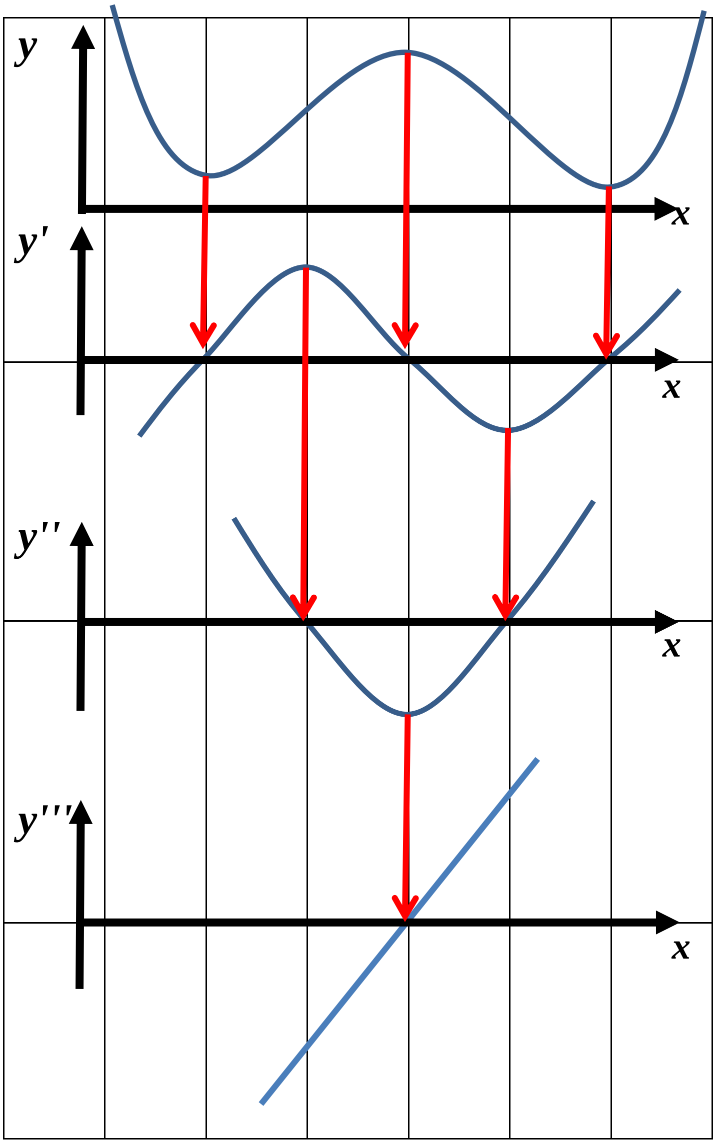
<table>
<tr><td>Typical Cubic Equation
$y = x^3 - 3x^2 = x^2(x - 3)$</td><td>Type of curve : Typical CUBIC
</td></tr>
<tr><td>Derivative
$\frac{dy}{dx} = 3x^2 - 6x = 3x(x - 2)$</td><td>Type of curve : Parabola
</td></tr>
<tr><td colspan="2">If we differentiate any CUBIC equation it becomes a QUADRATIC equation.</td></tr>
</table>

Typical QUARTIC Equation $y = x^2(x-2)^2 = x^4 - 4x^3 + 4x^2$	Type of curve : Typical Quartic 
Derivative $\dfrac{dy}{dx} = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2)$ $= 4x(x-1)(x-2)$	Type of curve : Cubic 
So if we differentiate a **QUARTIC** Equation it becomes a **CUBIC** equation.	

SUMMARISING:

Quadratic	**Cubic**	**Quartic**
 	 	

In the above graphs, the "idea" used is that **<u>points on the basic *x*, *y* graph where the gradient is zero,</u> mean that the <u>graph of the gradient function crosses the *x* axis at these same *x* values.</u>**
(See above how the vertical lines show this.)



A **QUARTIC** curve

$y = ax^4 + bx^3 + cx^2 + dx + e$

when differentiated becomes:

…a **CUBIC** curve

$y = fx^3 + gx^2 + hx + i$

when differentiated becomes:

…a **QUADRATIC** curve
(or **PARABOLA**)

$y = jx^2 \;\; kx + L$

when differentiated becomes:

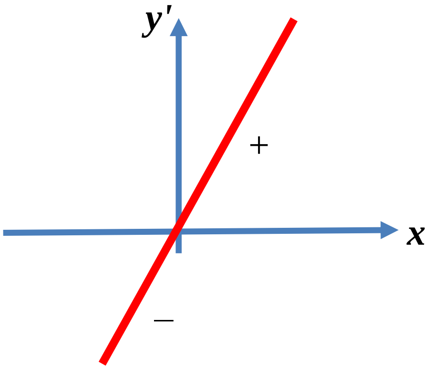
…a **LINE** graph

$y = mx + c$

NOW CONSIDER THE **OPPOSITE PROCESS**:

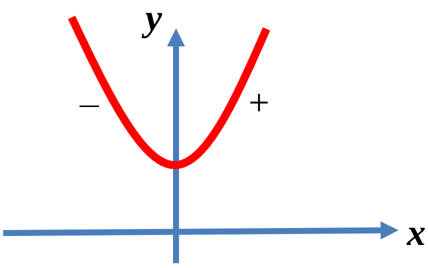
Given the graph of the DERIVATIVE, find the graph of the FUNCTION.

1. If we are given the graph $\frac{dy}{dx} = 2x$

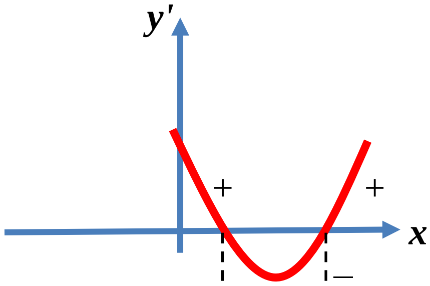


we know that the integral is $y = x^2 + c$

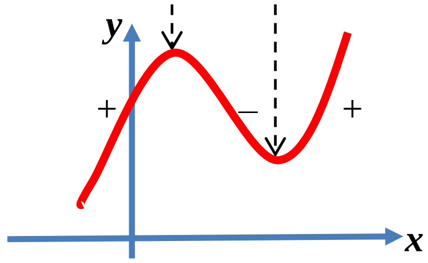
Unless we are given some condition such as "the function goes through the origin (0, 0)" then <u>we do not need to worry about the vertical position of the graph</u> (because "c" could be anything)



2. We do not need to be given the actual equation of $\frac{dy}{dx}$

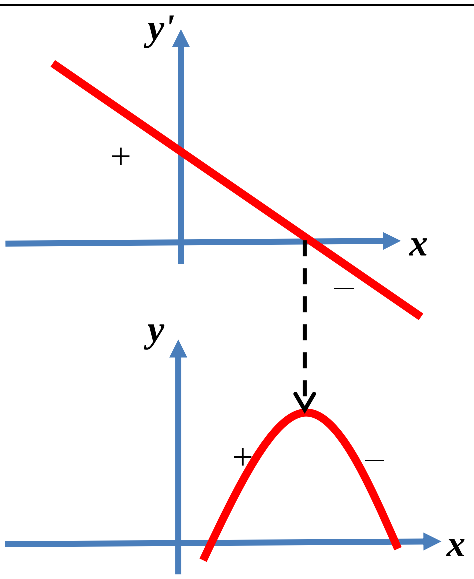


we can see that $y' = 0$ where the above graph crosses the x axis so we know the original function, y will have turning points at the same x values.



It is a good idea to draw it well away from the *x* axis.

3.



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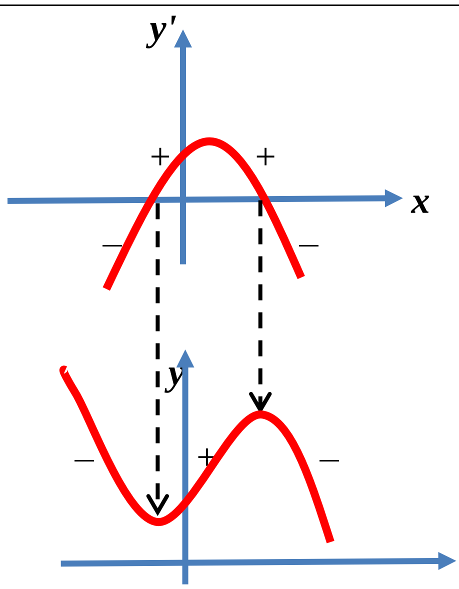
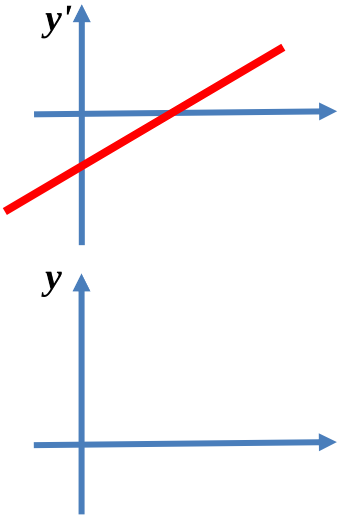
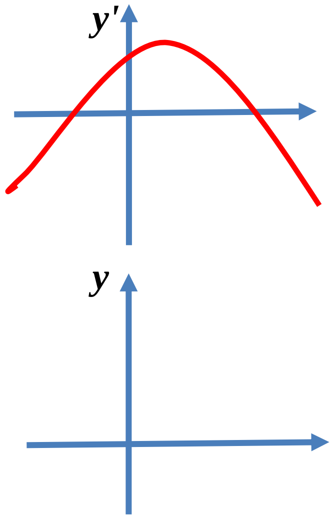
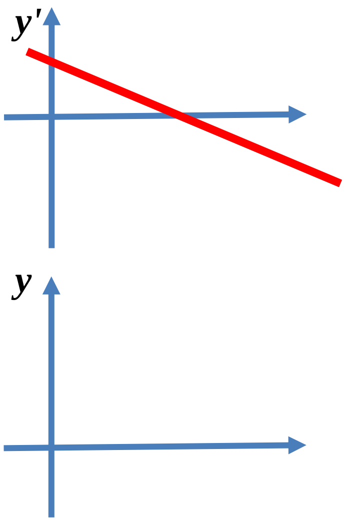
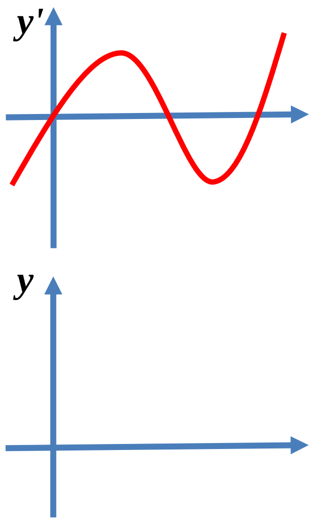
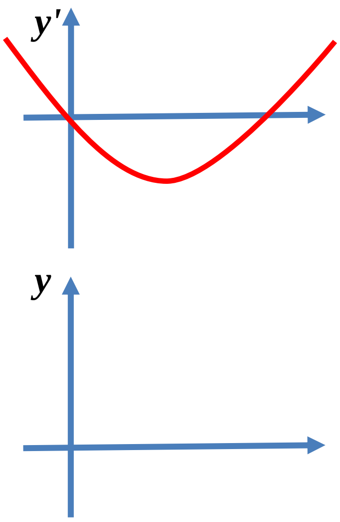
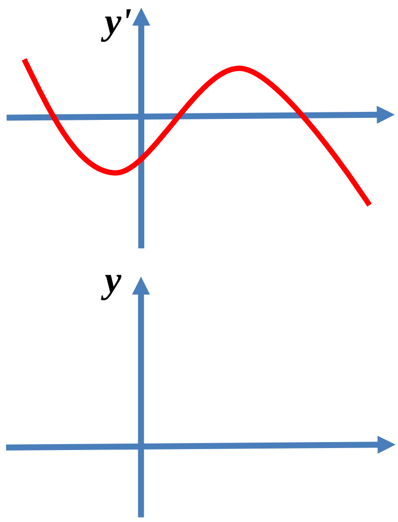
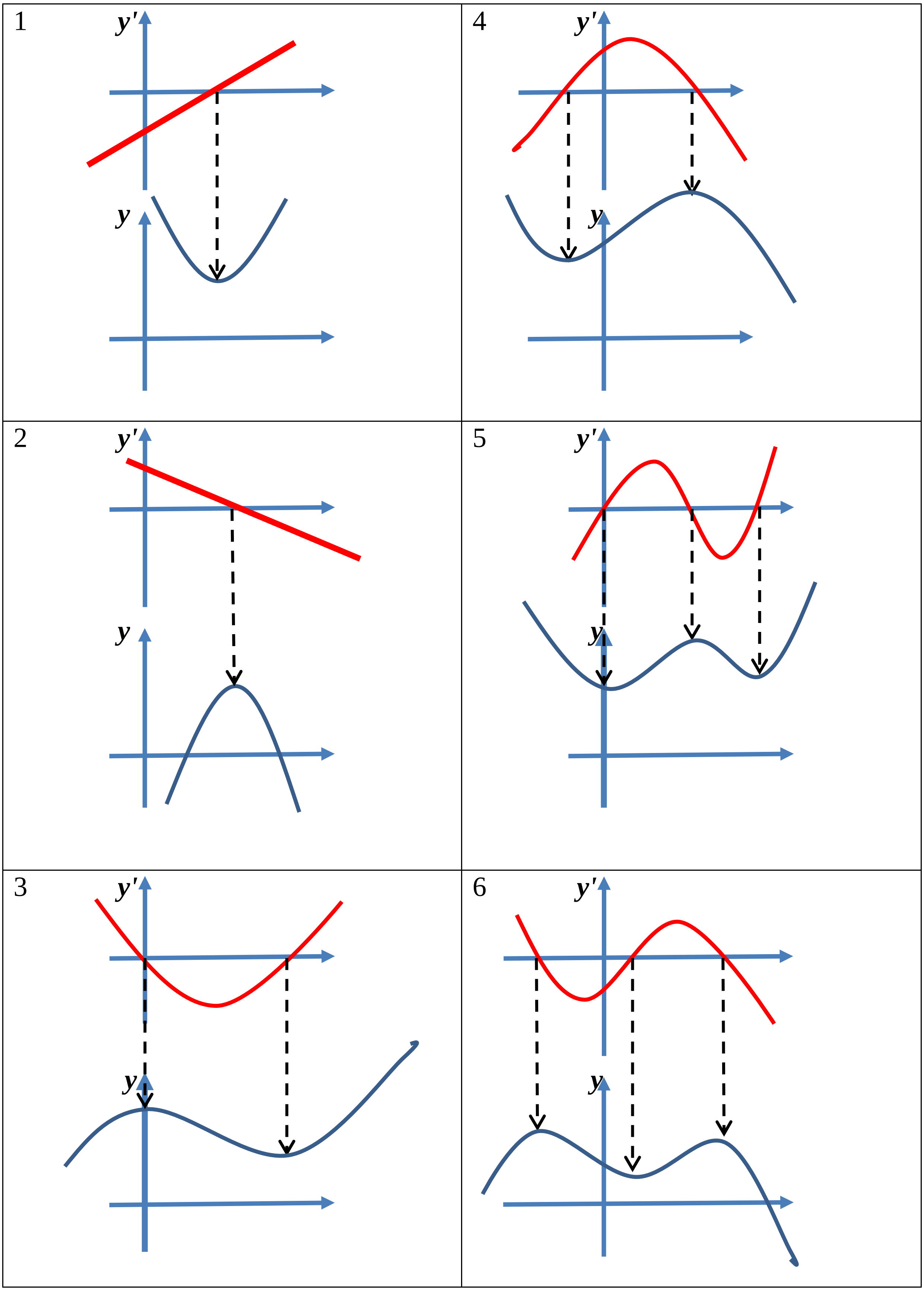


Given the following graphs of $\frac{dy}{dx}$, draw the graphs of the original functions.

1 	4 
2 	5 
3 	6 

Given the following graphs of $\dfrac{dy}{dx}$, draw the graphs of the original functions.

<u>SOLUTIONS</u>



ADVANCED SECTION:(Excellence Level)
THE THREE TYPES OF CUBIC GRAPHS.

These three **very similar** CUBIC EQUATIONS have **very different** GRAPHS:

$$y = x^3 - 3x^2 + 2x \;;\; y = x^3 - 3x^2 + 3x \;;\; y = x^3 - 3x^2 + 4x$$

---

1. $y = x^3 - 3x^2 + 2x$

$y' = 3x^2 - 6x + 2 = 0$ for max/min

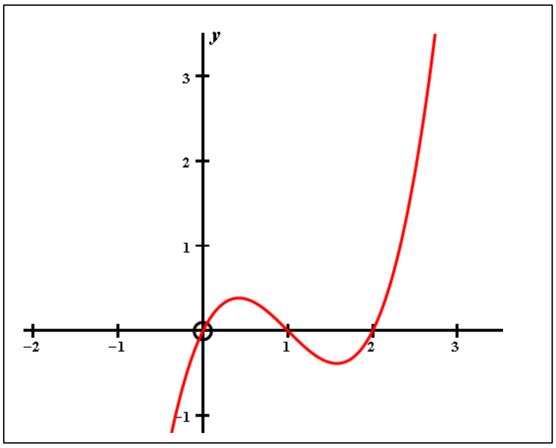
so x = 0.42 and 1.58

x	0.3	0.42	0.5		1.4	1.58	1.7
y ′	+	0	–		–	0	+

Max (0.42, 0.38) Min (1.58, – 0.38)

$y'' = 6x - 6$ = 0 at inflection point so x = 1

Inflection point at (1, 0)



2. $y = x^3 - 3x^2 + 3x$

If $y' = 3x^2 - 6x + 3 = 0$

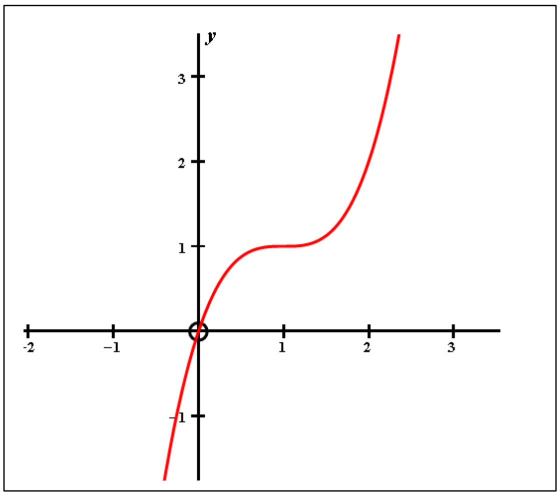
then x = 1 but it is neither a max nor a min **but the gradient is zero**.

x	.5	1	1.5
y ′	+	0	+

$y'' = 6x - 6$ = 0 at inflection point so x = 1

This cubic does not have a max or min point even though the gradient = 0

It has a **Stationary Inflection point** at (1, 1)



3. $y = x^3 - 3x^2 + 4x$

$y' = 3x^2 - 6x + 4$ = 0 for max/min

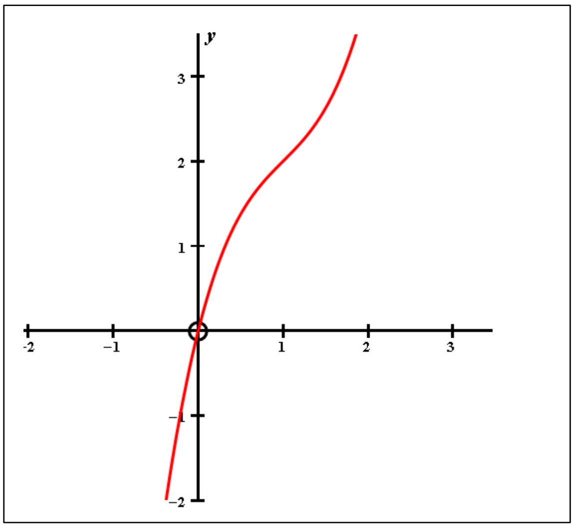
but this equation has **no real solutions**

so the gradient is NEVER zero.

x	0.5	1	1.5
y ′	1.75	1	1.75

$y'' = 6x - 6$ = 0 at inflection point so x = 1

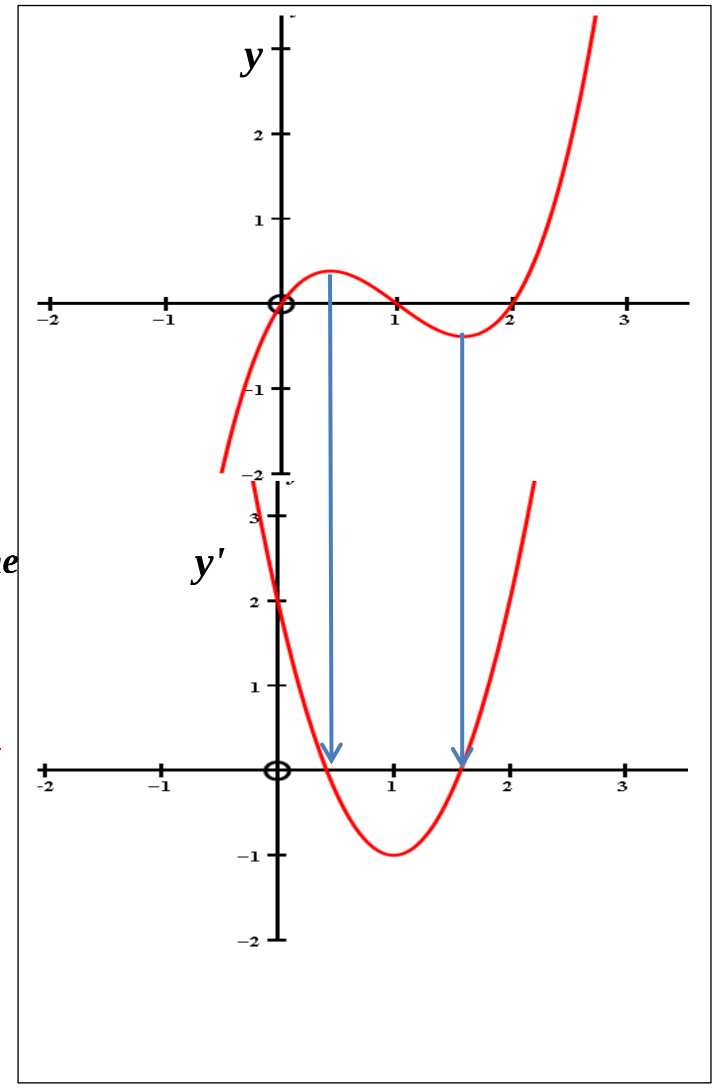
Inflection point is at (1, 2)



It is now very instructive to compare the above graphs with the graphs of their respective gradients.

1. ***Typical Cubic***

$$y = x^3 - 3x^2 + 2x$$
$$= x(x - 3x + 2)$$
$$= x(x - 1)(x - 2)$$



Gradient of cubic

$$y' = 3x^2 - 6x + 2$$

This is a parabola crossing the x axis at points where the gradient of the cubic is zero.

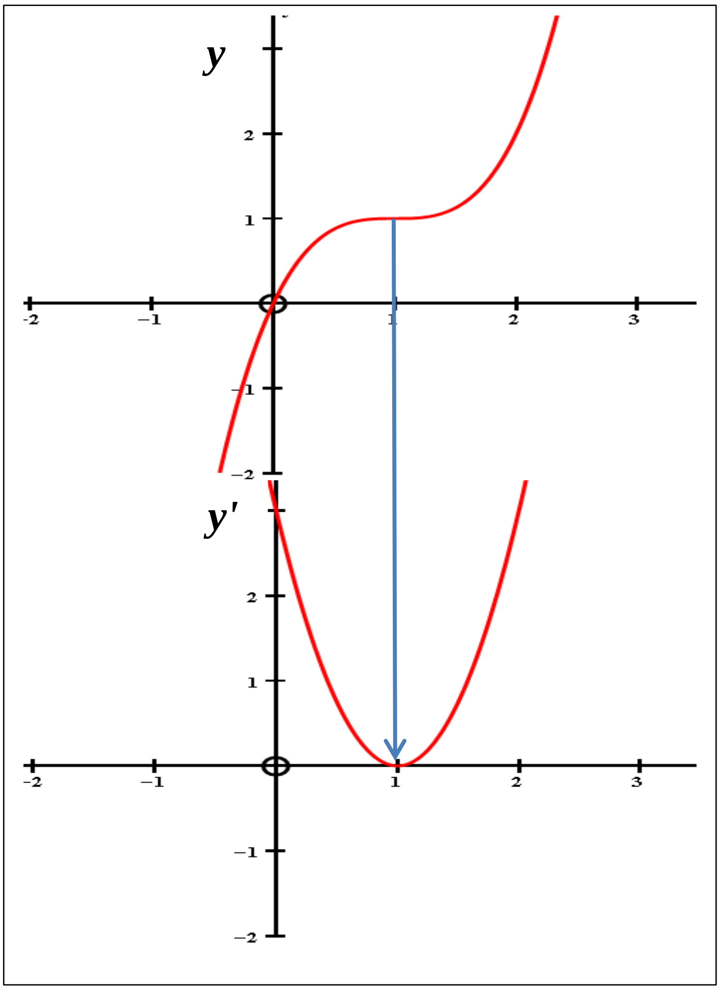
Notice the gradient of the cubic is zero at **two** points so the gradient graph crosses the ***x*** axis twice.

2. ***Special Cubic***

$y = x^3 - 3x^2 + 3x$

$= x(x^2 - 3x + 3)$

This cubic only crosses the x axis once at x = 0 because the other factor $(x^2 - 3x + 3) \neq 0$ (ie it has no real solutions)



$y' = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x - 1)^2$

The gradient is zero only at the one point x = 1

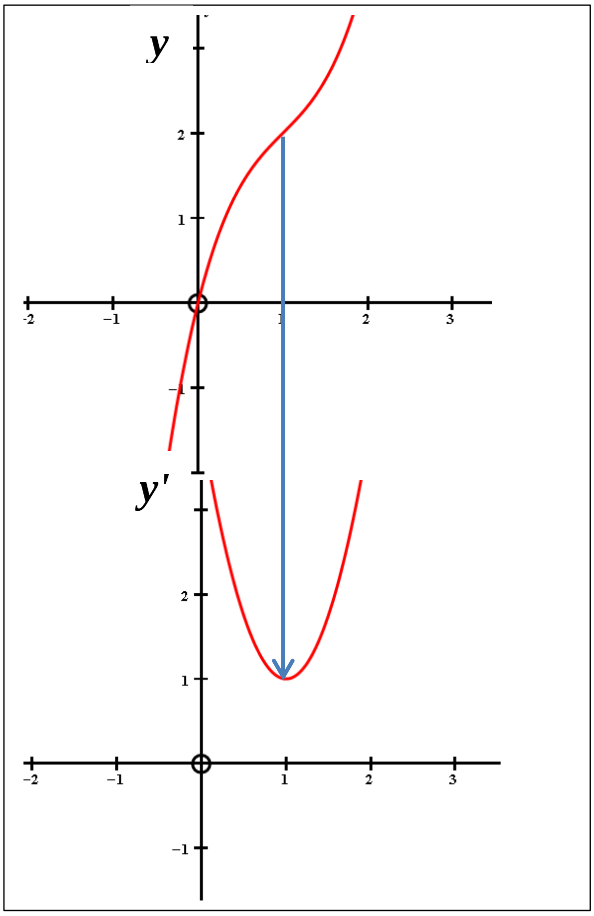
This is a parabola crossing the x axis at the only point where the gradient of the cubic is zero.

3. ***Another special Cubic***

$y = x^3 - 3x^2 + 4x$

$= x(x^2 - 3x + 4)$

This cubic only crosses the x axis once at x = 0 because the other factor $(x^2 - 3x + 4) \neq 0$ (ie it has no real solutions)

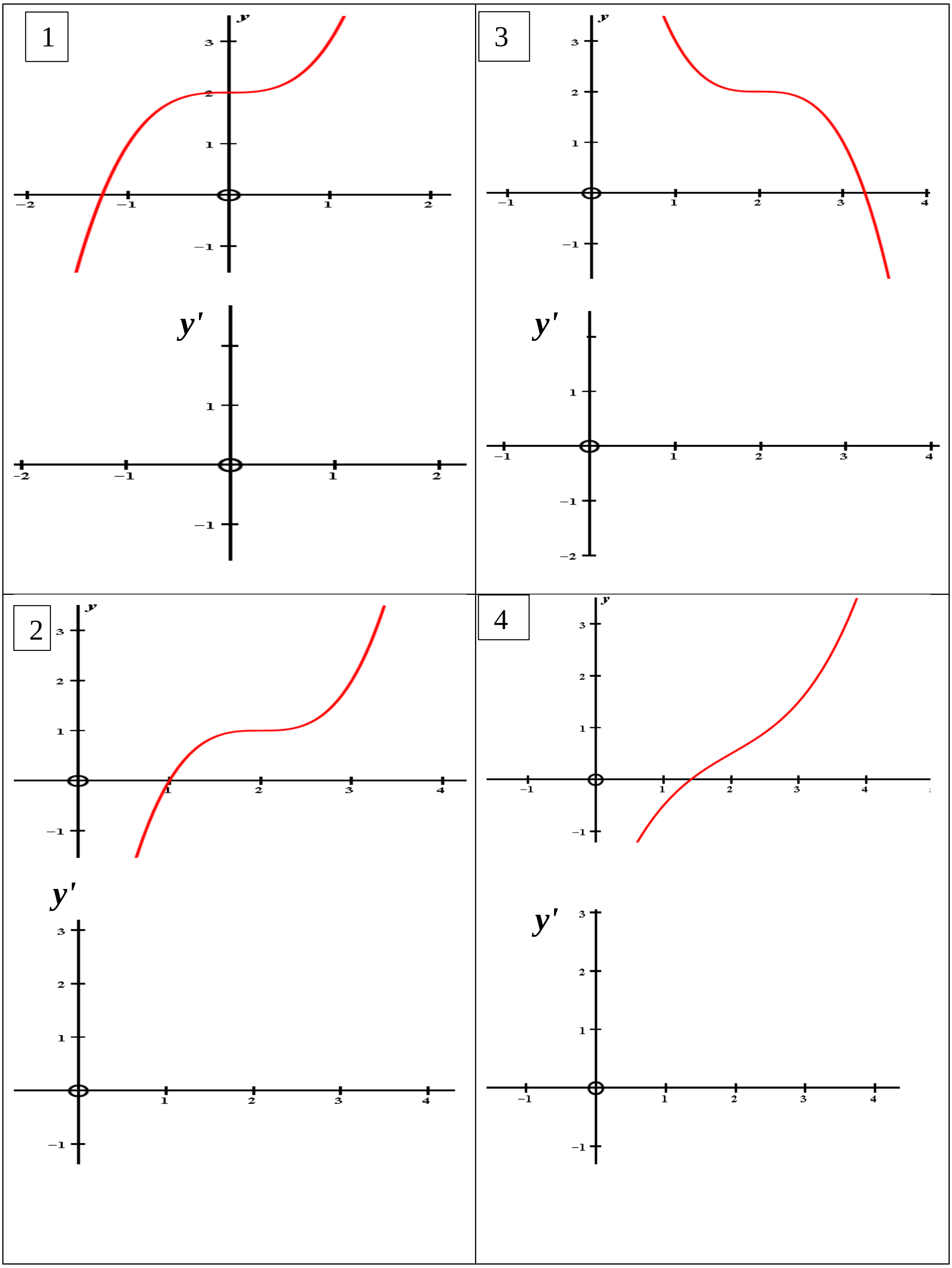


$y' = 3x^2 - 6x + 4$

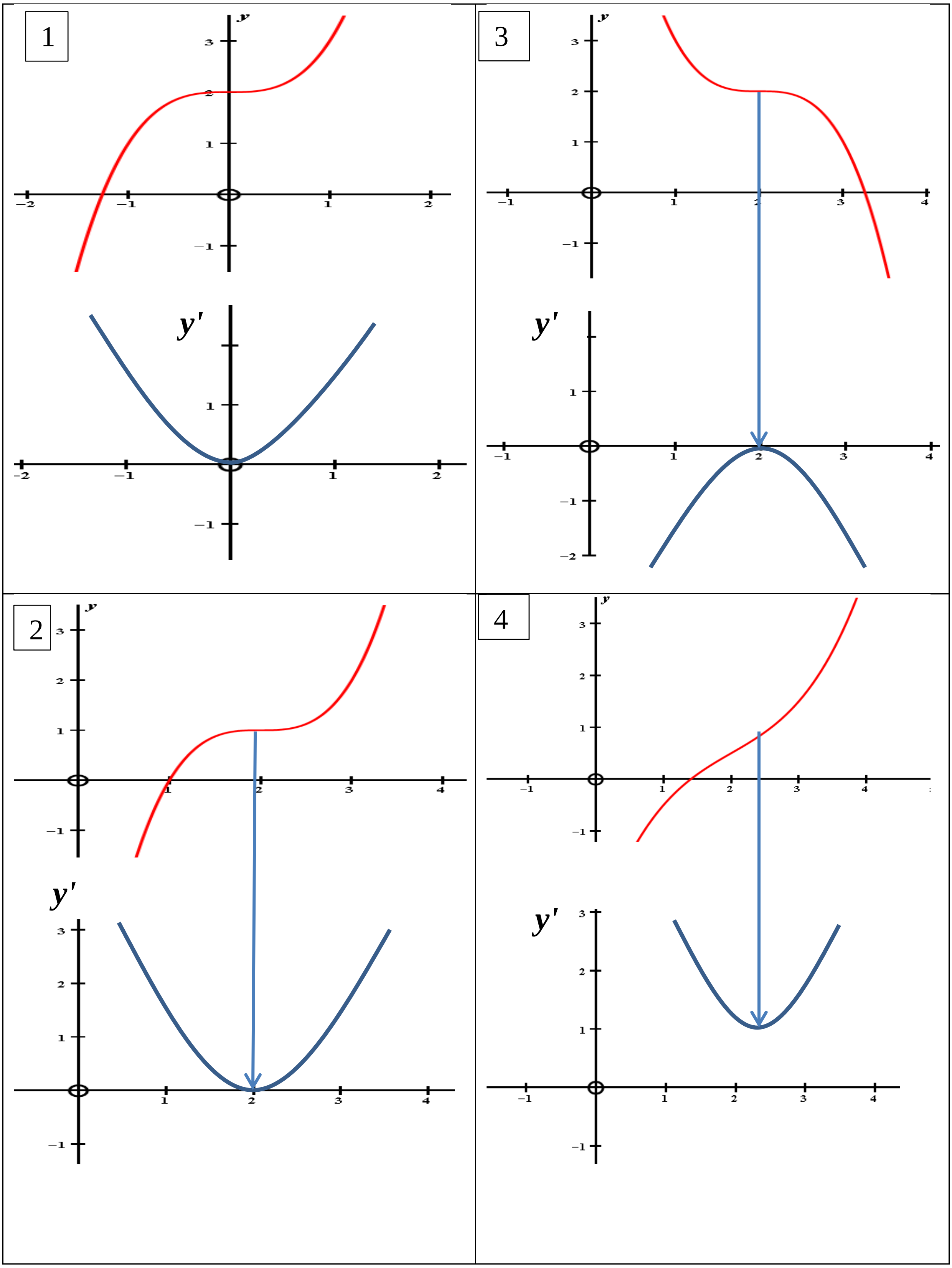
This quadratic has no real solutions so the graph does not cross the *x* axis.

This means that the gradient of the Cubic is never equal to zero.

Given the following graphs, draw the graphs of the GRADIENT functions.

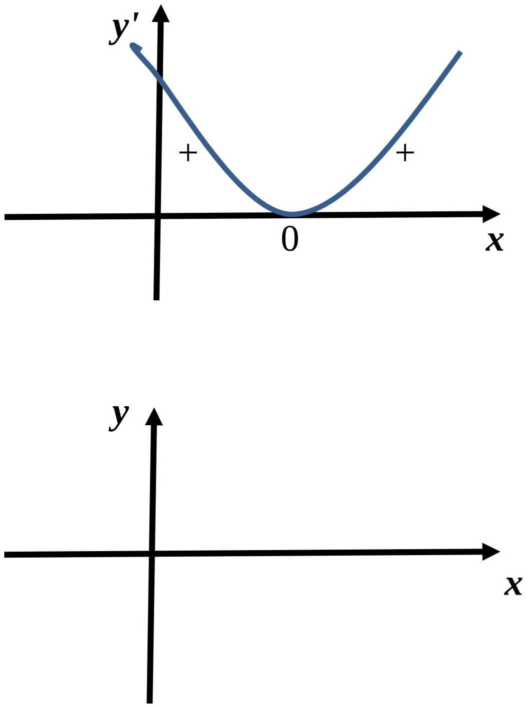


SOLUTIONS Given the following graphs, draw the graphs of the gradient functions .



Given the following **GRADIENT** graphs draw the **ORIGINAL FUNCTIONS**.

1.



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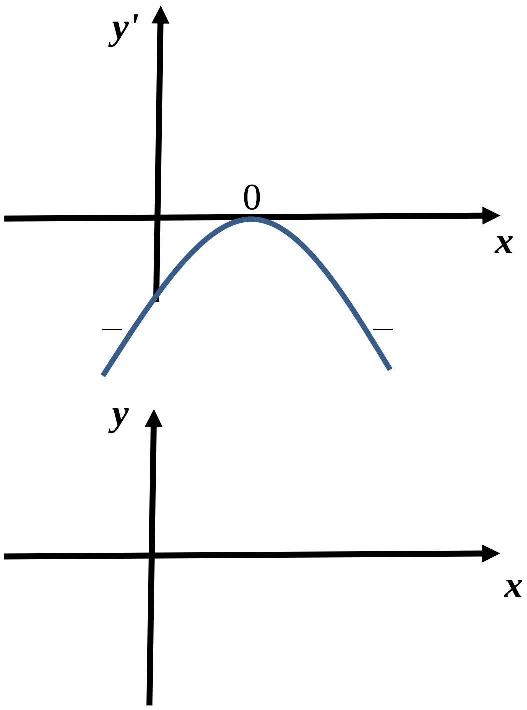

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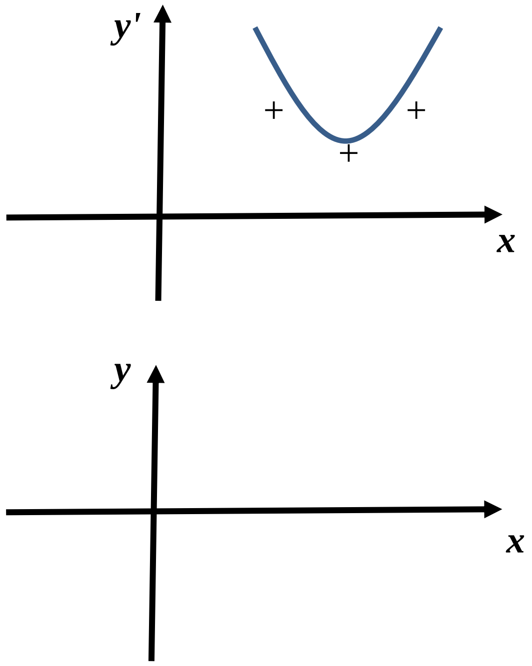

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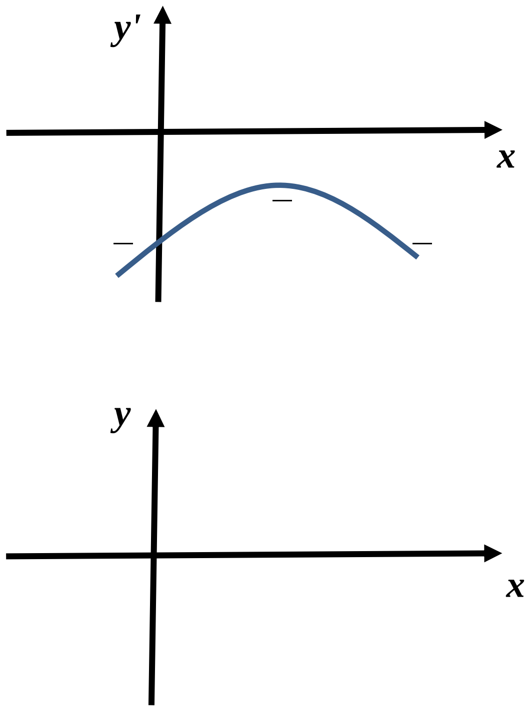

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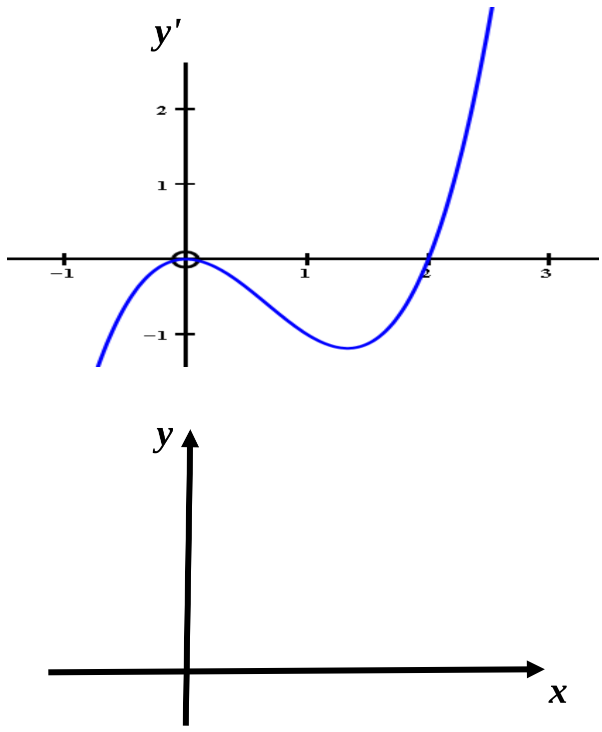

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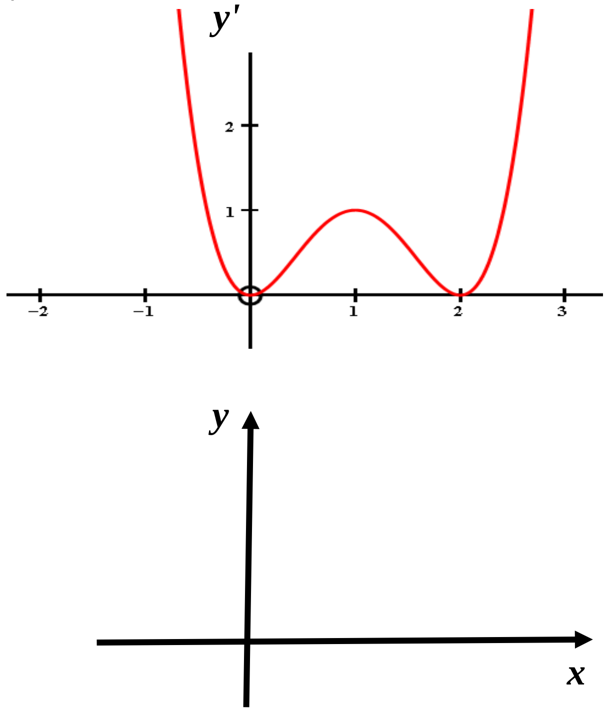

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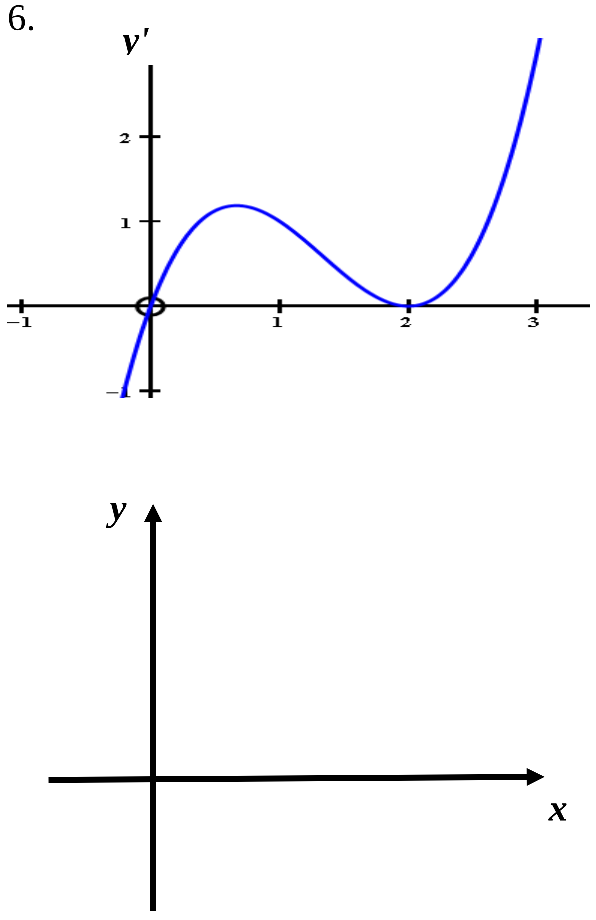

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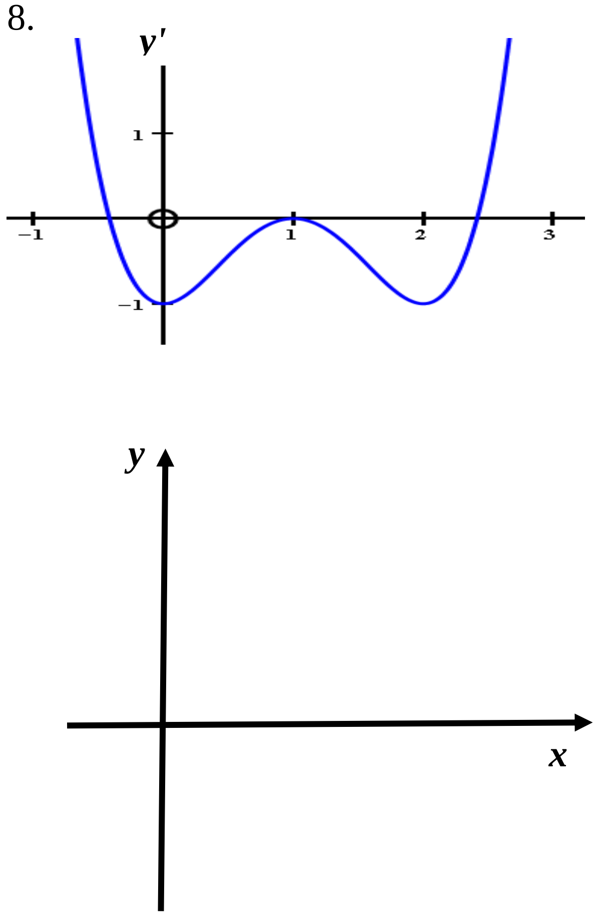
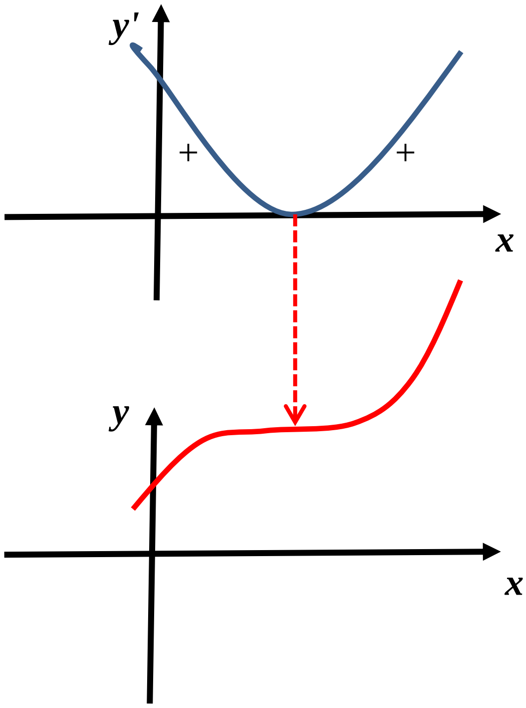


SOLUTIONS Given the following gradient graphs draw the original functions.

1.



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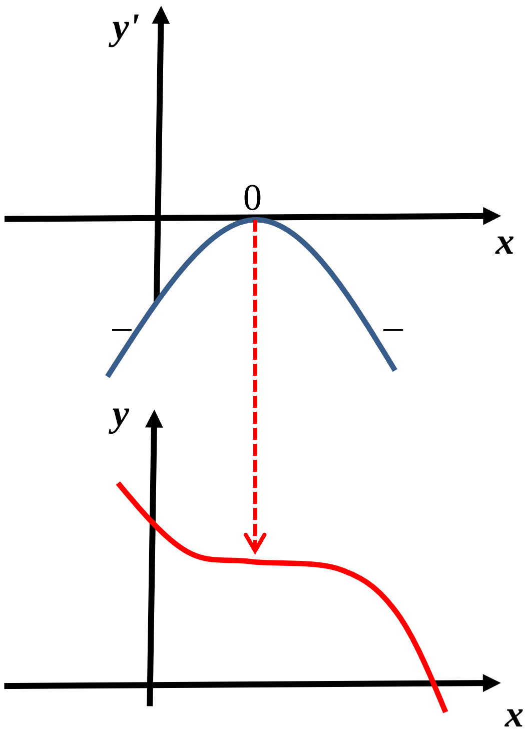

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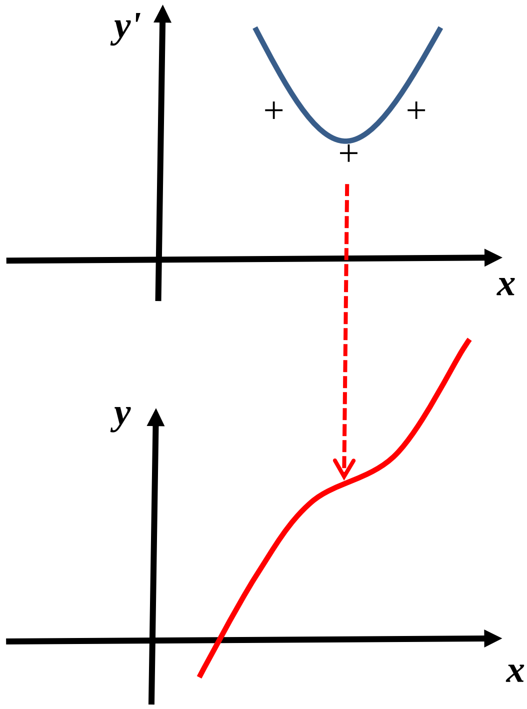

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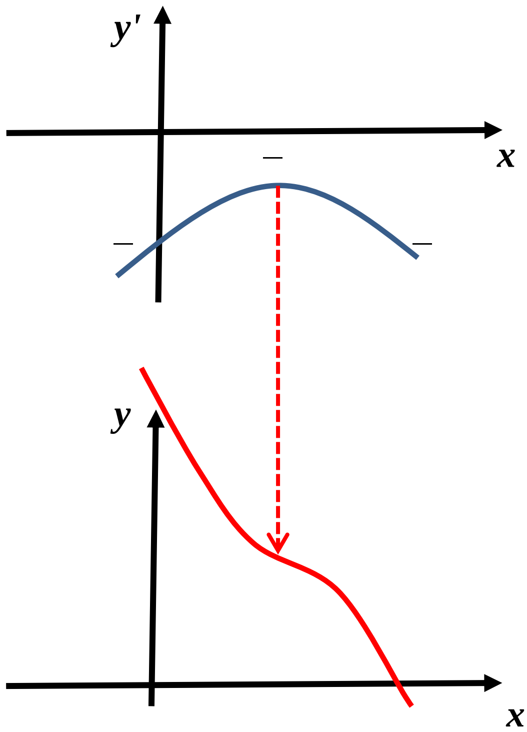

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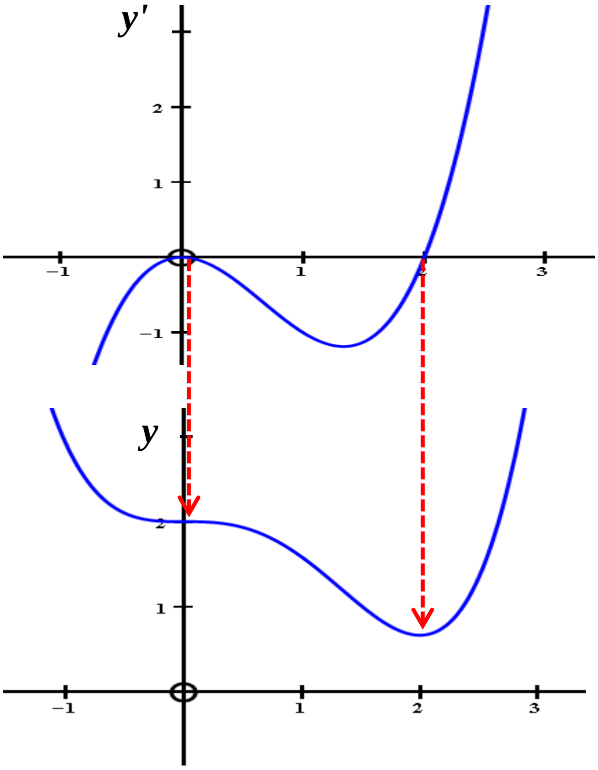
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3
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y
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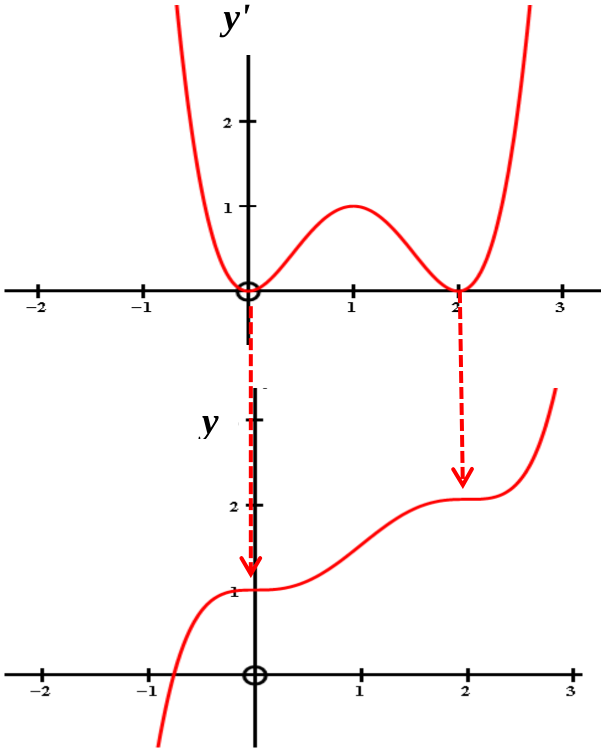
1
−1
1
2
3

7


y'
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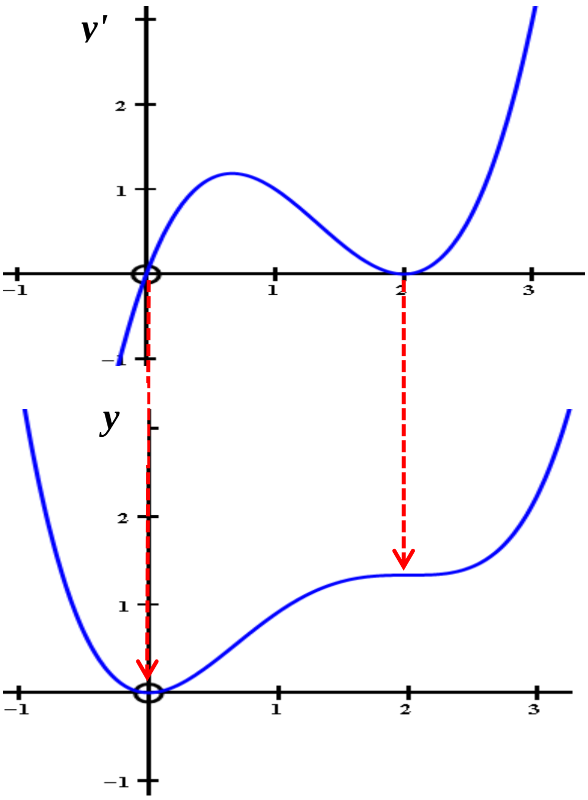
6.


y'
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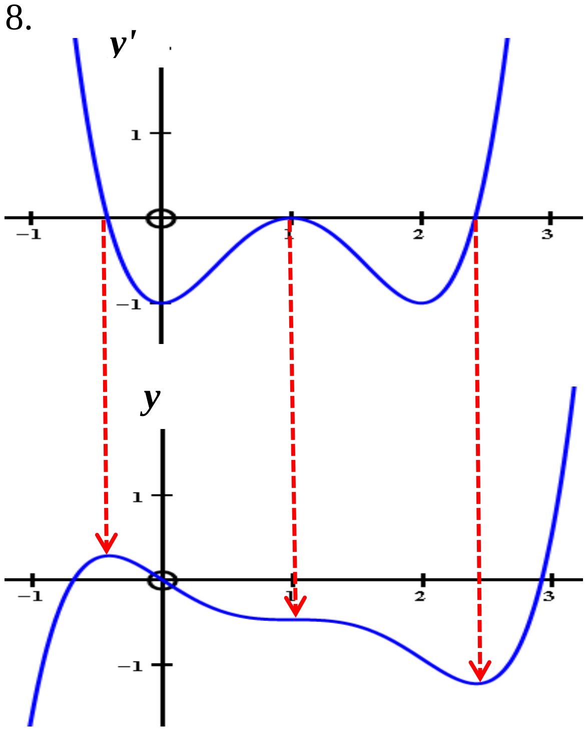
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