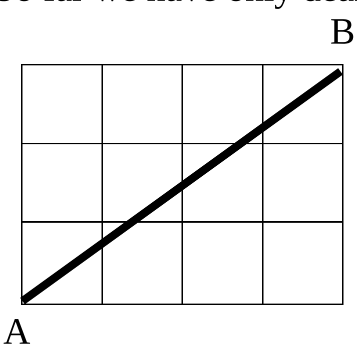
YEAR 12 A SHORT INTRODUCTION TO CALCULUS.

1. The ideas of Calculus were discovered at the same time by NEWTON in England and LEIBNITZ in Germany in the 17^{th} century. There was a lot of ill feeling between them because each one wanted to take the credit for discovering Calculus.

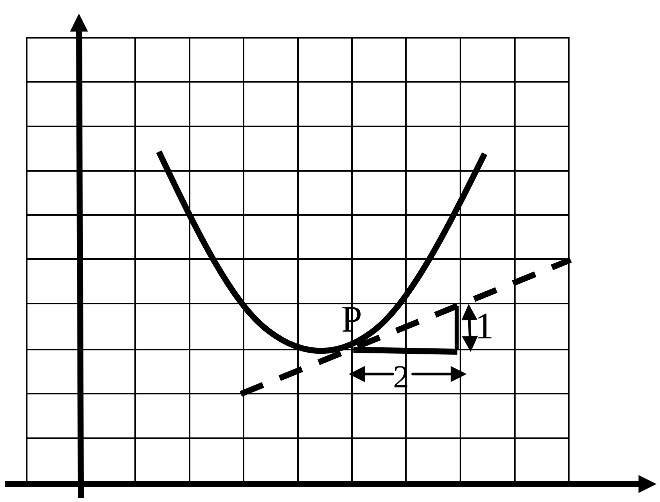
2. Newton had a particular interest in the orbits of planets and gravity. He "invented" calculus to help him study such topics.
   His theory was used extensively in putting the first men on the moon and his equations of motion clearly describe the paths of objects thrown through the air.
   Calculus can be applied to many subjects: finding equations to model the growth of animals, plants or bacteria; finding maximum profits in economics; finding the least amount of material to make boxes and cylinders; all sorts of velocity and acceleration problems.

3. The very basic idea of calculus is how to find the changing steepness of curves.
   So far we have only dealt with the gradients of lines.



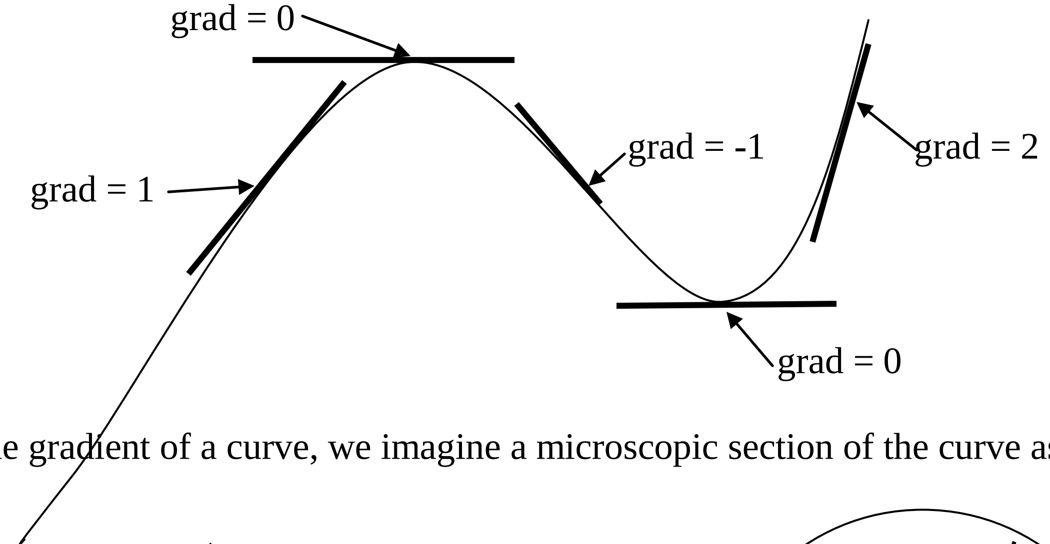
Gradient of AB = $\frac{3}{4}$

4. Definition: The steepness of a curve at a point P, is the steepness of the TANGENT to the curve at P.

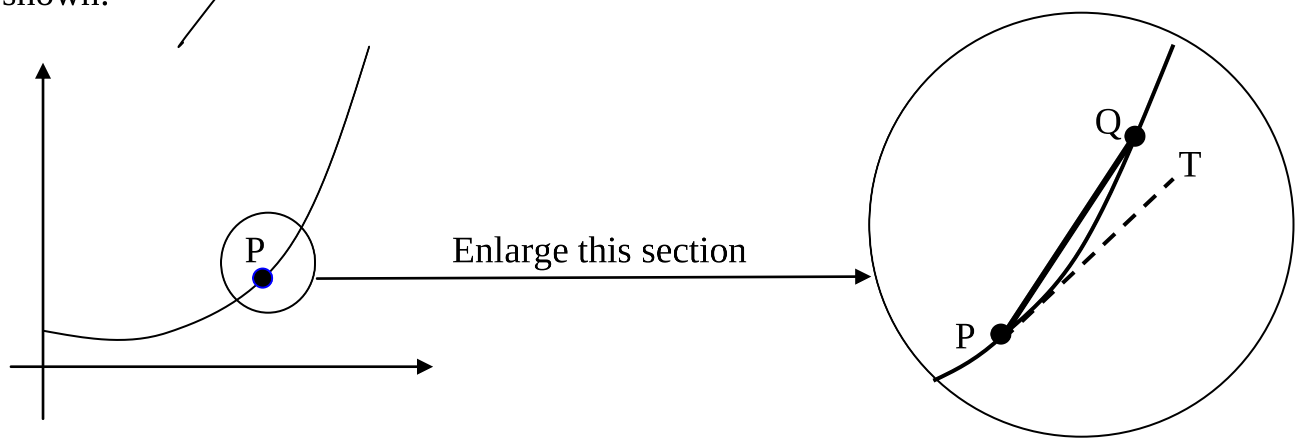


We can see that the gradient of the tangent to the curve, drawn at the point P(5, 3) is clearly $\frac{1}{2}$

5. The **approximate** values of the steepness (or gradient) of the following curve at various points are marked on this diagram:



6. To find the gradient of a curve, we imagine a microscopic section of the curve as shown:



<u>Fundamental Idea</u>: The steepness of tangent PT is approximately equal to the steepness of a chord PQ (where Q is a point on the curve very close to P).

This approximation gets closer to the gradient of the tangent at P by making Q move closer and closer to P.

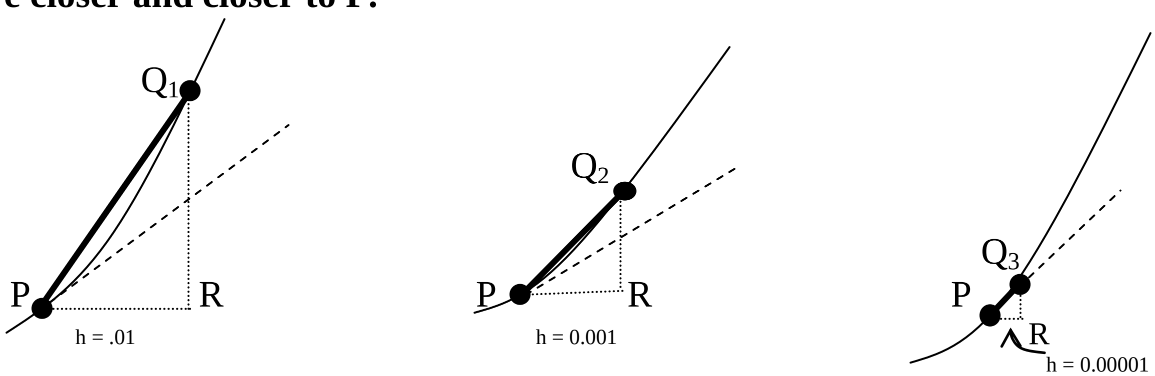


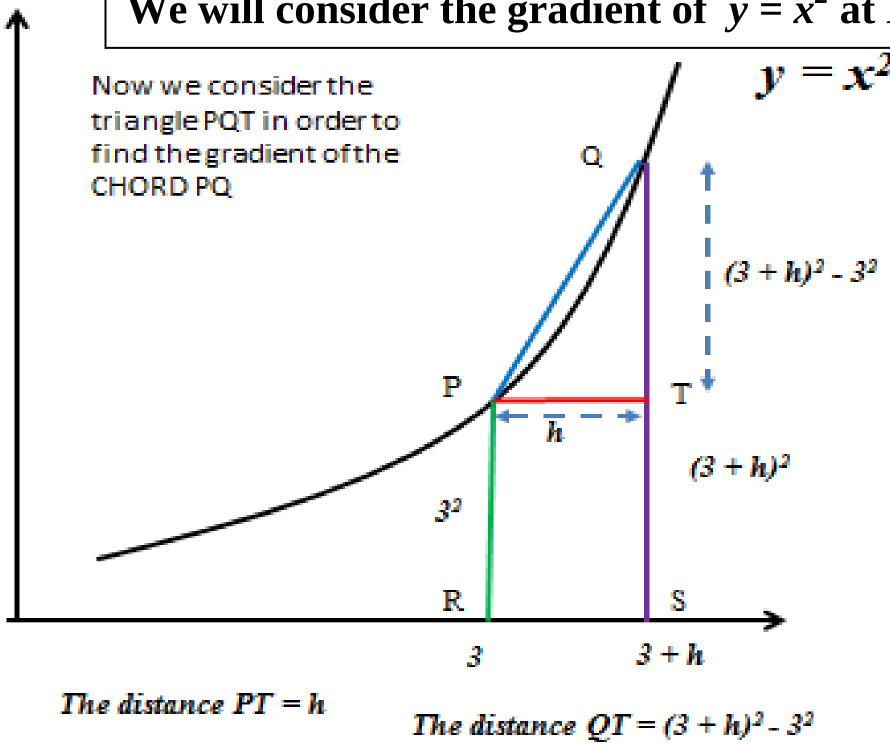
Fig 1. $\text{Grad } PQ_1 = \dfrac{Q_1R}{PR}$

Fig 2. $\text{Grad } PQ_2 = \dfrac{Q_2R}{PR}$

Fig 3. $\text{Grad } PQ_3 = \dfrac{Q_3R}{PR}$

This geometrical idea has to be changed into mathematical symbols so that we can find the numerical values of these gradients and work out what value they are approaching.

We will consider the gradient of $y = x^2$ at P where $x = 3$



The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(3 + h)^2 - 3^2}{h}$$

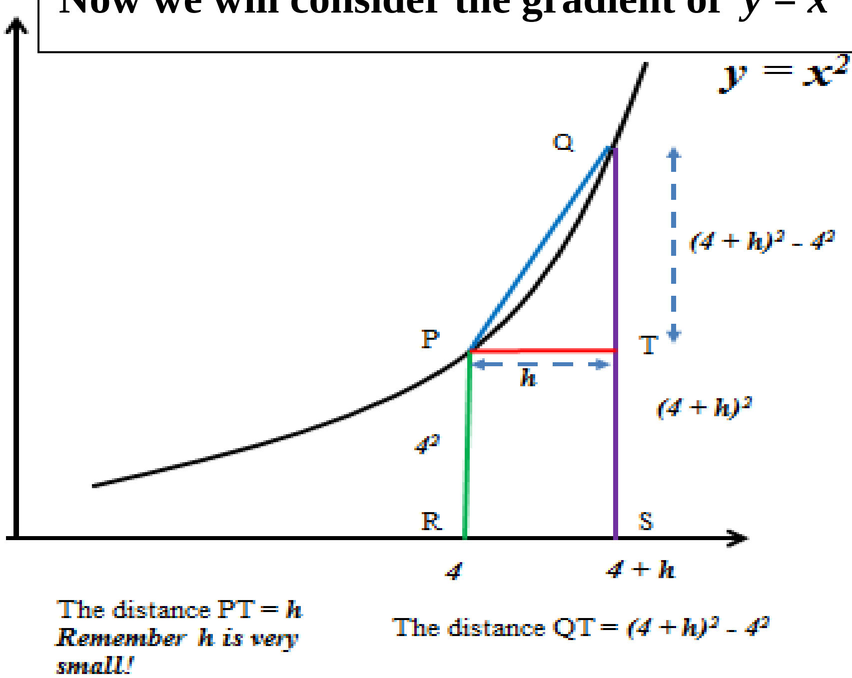
$$= \frac{9 + 6h + h^2 - 9}{h}$$

$$= \frac{h(6 + h)}{h}$$

$$= 6 + h$$

And when h reduces to 0 then the gradient is equal to 6

Now we will consider the gradient of $y = x^2$ at P where $x = 4$



The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(4 + h)^2 - 4^2}{h}$$

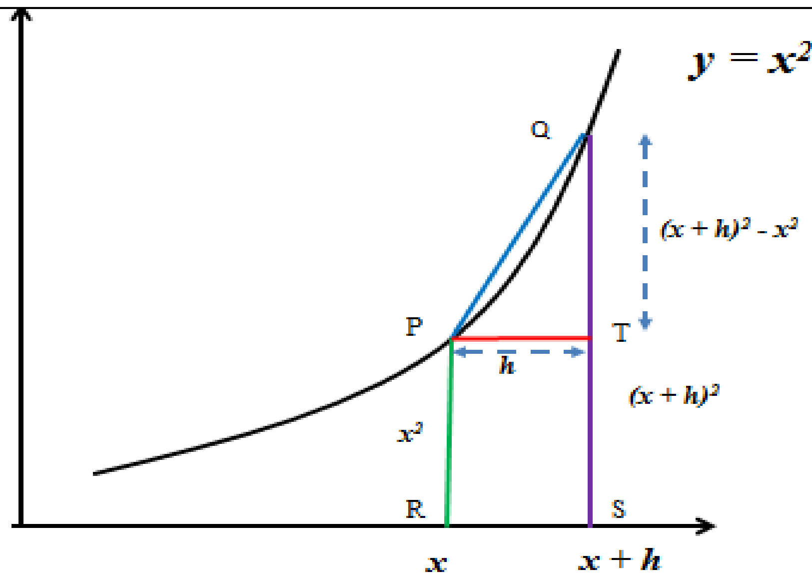
$$= \frac{16 + 8h + h^2 - 16}{h}$$

$$= \frac{h(8 + h)}{h}$$

$$= 8 + h$$

And when h reduces to 0 then the gradient is equal to 8

There is definitely a pattern here but to see it more clearly we will find the gradient at a general place "*x*" instead of specific values such as ***x = 3 or x = 4.***



The distance PT = h
Remember h is very small!

The distance QT = $(x+h)^2 - x^2$

The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{h(2x+h)}{h}$$

$$= 2x + h$$

And when h reduces to 0 then the gradient is equal to $2x$

The main symbol we use for the gradient of a curve is y' (pronounced "***y dash***")

We have just found that the gradient of the curve

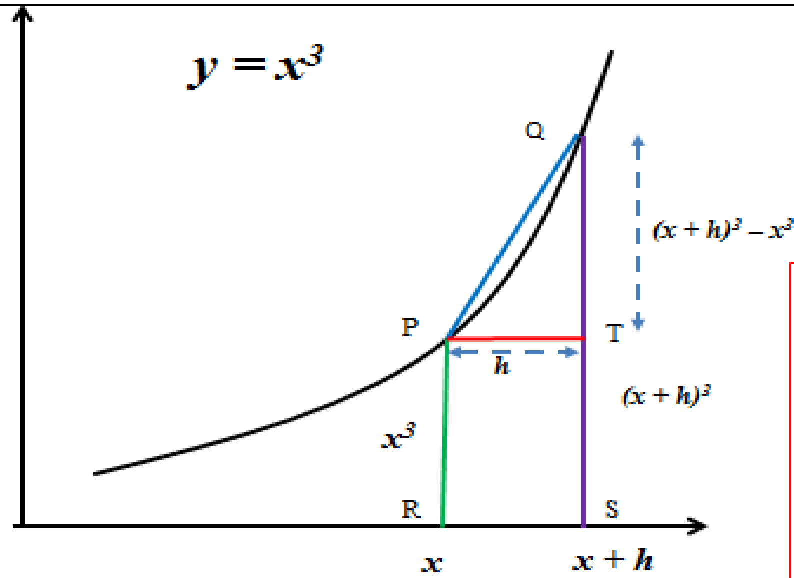
$y = x^2$ at any point x is $y' = 2x$

This means that for the curve $y = x^2$ **:**

at $x = 1$, the gradient $y' = 2 \times 1 = 2$
at $x = 2$, the gradient $y' = 2 \times 2 = 4$
at $x = 3$, the gradient $y' = 2 \times 3 = 6$
at $x = 4$, the gradient $y' = 2 \times 4 = 8$
at $x = 10$, the gradient $y' = 2 \times 10 = 20$
at $x = -6$, the gradient $y' = 2 \times (-6) = -12$
at $x = ½$, the gradient $y' = 2 \times ½ = 1$

We found a simple pattern for $y = x^2$ but there is also a pattern for any power of x.

Let us consider the curve $y = x^3$ and find the gradient at a general point "x"



The distance PT = h
Remember h is very small!

The distance QT = $(x + h)^3 - x^3$

The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(x + h)^3 - x^3}{h}$$

We need more room to work this out.

The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(x + h)^3 - x^3}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

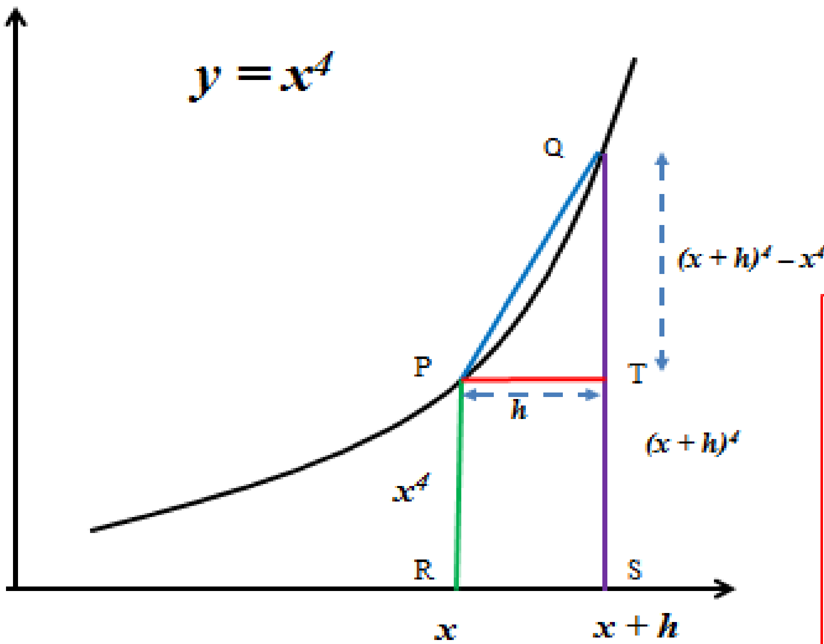
$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= 3x^2 + 3xh + h^2$$

$= 3x^2$ *when h reduces to zero*

We will repeat this process for the curve $y = x^4$ then the pattern will be obvious to everybody!



The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(x+h)^4 - x^4}{h}$$

We need more room to work this out.

The distance PT = h

***Remember** h is very small!*

The distance QT = $(x + h)^4 - x^4$

The gradient of PQ

$$= \frac{QT}{PT}$$

$$= \frac{(x+h)^4 - x^4}{h}$$

$$= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$$

$$= 4x^3 + 6x^2h + 4xh^2 + h^3$$

$= 4x^3$ ***when h reduces to zero***

CONCLUSION!

$If\ y = x^2\ the\ gradient\ is\ y' = 2x^1$

$If\ y = x^3\ the\ gradient\ is\ y' = 3x^2$

$If\ y = x^4\ the\ gradient\ is\ y' = 4x^3$

$If\ y = x^5\ the\ gradient\ is\ y' = 5x^4$

$If\ y = x^6\ the\ gradient\ is\ y' = 6x^5$

$$If\ y = x^n\ the\ gradient\ is\ y' = n \times x^{(n-1)}$$

We could repeat the theory to be absolutely sure, but I think we can easily accept the following:

$If\ y = 5x^2\ then\ y' = 2 \times 5x^1 = 10x$

$If\ y = 7x^3\ then\ y' = 21x^2$

$If\ y = 3x^5\ then\ y' = 15x^4$

$If\ y = 2x^7\ then\ y' = 14x^6$

$$Generally\ if\ y = ax^n\ then\ y' = nax^{(n-1)}$$

SPECIAL NOTES:

The equation $y = 3x$ represents a line graph and we already know that its gradient is 3 so we could write $y' = 3$.

Interestingly this also fits the pattern:

We "could" say $y = 3x^1$ so $y' = 1 \times 3 \times x^0 = 3$

Similarly, the equation $y = 4$ represents a horizontal line and we already know that its gradient is zero.

The process of finding the gradient is called DIFFERENTIATION.

When we have an equation with several terms such as:

$$y = 3x^5 + 6x^4 + 2x^3 + 5x^2 + 7x + 9$$

...it is a good idea to treat each term as a separate bit and we just apply the general rule to each term in turn.

$$\textbf{If} \quad y = 3x^5 + 6x^4 + 2x^3 + 5x^2 + 7x + 9$$

$$\textbf{then } y' = 15x^4 + 24x^3 + 6x^2 + 10x + 7 + 0$$

Usually we would just write the following:

Question.

Differentiate the function $y = x^3 - 5x^2 + 3x + 2$

Answer: $y' = 3x^2 - 10x + 3$