## Teaching notes on RATE OF CHANGE Y12/13

1. A balloon is being pumped up so that its volume, $\boldsymbol{V}$, in $\mathrm{cm}^{3}$ at t seconds is $V=2 t+5$


The Graph of $\boldsymbol{V}$ is :


Gradient of this graph is 2
Conclusion:
The RATE of increase of the Volume $=$ the Gradient of the graph
Or, we can find the RATE of INCREASE by DIFFERENTIATING.
Rate of Increase of $V=\frac{d V}{d t}$
2.(a) Suppose a drop of water falls on a paper towel and the circular wet patch increases in radius according to the equation $r=3 t+4$
where $t$ is in seconds and $r$ is the radius in mm .
The rate of increase of the radius is : $\frac{d r}{d t}=3 \mathrm{~mm} / \mathrm{sec}$
(b) The rate of increase of the Circumference can be found from:

$$
\begin{aligned}
C & =\pi D \\
& =\pi 2 r \\
& =\pi 2(3 t+4) \\
& =8 \pi+6 \pi t
\end{aligned}
$$

Rate of $\mathrm{Inc}=\underline{d C}=0+6 \pi \approx 18.8 \mathrm{~mm} / \mathrm{sec}$
(c) The rate of increase of the AREA of the wet patch can be found from:

$$
\begin{aligned}
A & =\pi r^{2} \\
& =\pi(3 t+4)^{2} \\
& =\pi\left(9 t^{2}+24 t+16\right) \\
& =9 \pi t^{2}+24 \pi t+16 \pi
\end{aligned}
$$

Rate of increase is:

$$
\frac{d A}{d t}=18 \pi t+24 \pi
$$

This time, the rate of increase is not a constant value but depends on the time $t$.

| $\boldsymbol{t}$ secs | Rate of increase of <br> area $=\frac{d \boldsymbol{d}}{\boldsymbol{d t}} \mathrm{~mm}^{2} / \mathbf{s e c}$ |
| :--- | :--- |
| 0 | $24 \pi$ |
| 1 | $24 \pi+18 \pi=42 \pi$ |
| 2 | $24 \pi+36 \pi=60 \pi$ |

3. Suppose a jogger is running steadily so that the distance covered, $\boldsymbol{x}$, is given by $\boldsymbol{x}=4 \boldsymbol{t}$ where $\boldsymbol{t}$ is the time in seconds.

| $\boldsymbol{t}$ | $\boldsymbol{x}$ |
| :---: | :---: |
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |
| 3 | 12 |

This is what we call the VELOCITY ( or loosely, the "speed") of the jogger.
Rate of increase of distance $=$ velocity $\quad O R \quad V=\frac{d x}{d t}$
4. Suppose a mechanical car moves so that its distance, s, from O at t sec is given by $s=t^{2}$

Then its VELOCITY is $V=\frac{d s}{d t}=2 t$

| $\boldsymbol{t}$ | $\boldsymbol{s}=\boldsymbol{t}^{\mathbf{2}}$ | $\boldsymbol{v}=\mathbf{2 t}$ |
| :---: | :---: | :---: |
| 0 | 0 | 0 |
| 1 | 1 | 2 |
| 2 | 4 | 4 |
| 3 | 9 | 6 | The Velocity is increasing at a rate of $2 \mathrm{~m} / \mathrm{sec}$ every second

This is often confusingly written as :
$2 \mathrm{~m} / \mathrm{sec} / \mathrm{sec}$
or $2 \mathrm{~m} / \mathrm{sec}^{2}$
or $2 \mathrm{~m} . \mathrm{sec}^{-2}$
(whatever a "square second" is ! )

The Rate of increase of the Velocity is what we call ACCELERATION so accel $a=\frac{d v}{d t}=2 \mathrm{~m} / \mathrm{s}$ every sec or $2 \mathrm{~m} / \mathrm{s} / \mathrm{s}$
5. Suppose a model car moves so that its distance s , from O , at t sec is: $s=t^{2}+3 t+2$
Clearly : velocity $V=\frac{d s}{d t}=2 t+3 \mathrm{~m} / \mathrm{s}$

$$
\text { and acceleration } a=\frac{d V}{d t}=2 \quad m / s \text { every sec or } m / s / s
$$

6. A ball is kicked vertically up into the air so that its height, $h$ metres, at t sec is :

$$
h=30 t-5 t^{2}
$$

The velocity equation is $V=\frac{d \boldsymbol{h}}{\boldsymbol{d} \boldsymbol{t}}=\mathbf{3 0}-\mathbf{1 0 t}$
and the acceleration equation is $a=\frac{d V}{d t}=-10 \begin{gathered}\text { (this is the acceleration } \\ \text { due to gravity) }\end{gathered}$
Let us work out values of $\boldsymbol{h}$ and $\boldsymbol{V}$ for various times.

| $\boldsymbol{t}$ sec | $\boldsymbol{h}$ metres | $\boldsymbol{V} \mathbf{m} / \mathbf{s}$ |
| :---: | :---: | :---: |
| 0 | 0 | 30 |
| 1 | 25 | 20 |
| 2 | 40 | 10 |
| 3 | 45 | 0 |
| 4 | 40 | -10 |
| 5 | 25 | -20 |
| 6 | 0 | -30 |

Clearly, the ball travels up a maximum height of 45 m and reaches that height at $\mathrm{t}=3$ sec. It then starts to come down again (at the same speed that it went up - which is what the negative velocity means.)
The above information can be found without writing out the above table:
Eg At the highest point, the Velocity is Zero so $\mathbf{3 0} \mathbf{- 1 0 t}=\mathbf{0}$

$$
\begin{array}{ll}
\text { so } & 30=10 t \\
\text { giving } & t=3 \mathrm{sec}
\end{array}
$$

The Max height is the $h$ value when $t=3$

$$
\text { so } \quad \begin{aligned}
h & =30 t-5 t^{2} \\
& =30 \times 3-5 \times 3^{2} \\
& =45 \text { metres }
\end{aligned}
$$

The INITIAL VELOCITY is the "velocity at the instant the ball is kicked".
This means, find V at $\mathrm{t}=0$
If $V=30-10 t$ then at $t=0 \quad V=30 \mathrm{~m} / \mathrm{s}$
7. A stone is thrown vertically up from the top of a cliff which is 25 metres high. The height of the stone at t secs from the top of the cliff is:
$h=20 t-5 t^{2}$
Find (a) the initial velocity of the stone
(b) the maximum height the stone reaches
(c) the time taken for the stone to reach the bottom of the cliff.
(d) the speed that the stone hits the beach at the bottom of the cliff.
(a) $V=20-10 t$
so if $t=0$
then $V=20 \mathrm{~m} / \mathrm{s}$
(b) The Max height is when the velocity $=0$

$$
h=20 t-5 t^{2}
$$

so $V=20-10 t=0$ at max height

$$
\text { so } \quad 20=10 t
$$

$$
t=2 \mathrm{sec}
$$

Max height is value of $h$ at $t=2$

$$
\begin{aligned}
& h=20 t-5 t^{2} \\
& h=20 \times 2-5 \times 2^{2} \\
& h=20 \text { metres }
\end{aligned}
$$

(c) At the bottom of the cliff, the value of $\boldsymbol{h}$ is -25

$$
\text { so } \begin{aligned}
20 t-5 t^{2} & =-25 \\
0 & =5 t^{2}-20 t-25 \\
0 & =5\left(t^{2}-4 t-5\right) \\
0 & =5(t+1)(t-5) \\
\text { so } t & =5 \text { sec (the solution } t=-1 \text { does not apply) }
\end{aligned}
$$

(d) At $\mathrm{t}=5$ the velocity is $\boldsymbol{V}=\mathbf{2 0} \mathbf{- 1 0 t}$

$$
\begin{aligned}
& =20-10 \times 5 \\
& =-30 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

The stone hits the beach at a speed of $30 \mathrm{~m} / \mathrm{s}$.
$8^{* * *}$. A mechanical car runs on a straight track. It starts from rest at point O and accelerates for a short time then slows down and comes to rest momentarily. It then repeats the motion in reverse, finally coming to rest at O after 6 seconds altogether. The equation which models this motion is

$$
x=t^{2}(t-6)^{2} \quad \text { where } x \text { is in } c m, t \text { is in sec }
$$

The graph of this function is:


We can effectively "cut off" parts of this graph because the model only applies for $\boldsymbol{t}$ values from 0 to 6.


We can now use Calculus to work out details of this motion:

$$
\begin{aligned}
x & =t^{2}(t-6)^{2} \\
& =t^{2}\left(t^{2}-12 t+36\right) \\
& =t^{4}-12 t^{3}+36 t^{2}
\end{aligned}
$$

so velocity equation is:

$$
\begin{aligned}
v=\frac{d x}{d t} & =4 t^{3}-36 t^{2}+72 t \\
& =4 t\left(t^{2}-9 t+18\right) \\
& =4 t(t-3)(t-6)
\end{aligned}
$$

Clearly, the velocity, $\boldsymbol{v}$, is zero at times $\mathrm{t}=0,3$ and 6 sec .

From the graph we can see that at $t=0$, the car is at the starting point O .
At $\mathrm{t}=3$, it is at its maximum distance from O so substituting $t=3$ in the distance equation gives:
Max dist $=3^{2}(3-6)^{2}=81 \mathrm{~cm}$ from 0 .
At $\mathrm{t}=6$, it is back at O .
If we consider the acceleration of the car, we get:
Accel $=\frac{d v}{d t}=12 t^{2}-72 t+72$ (this equation does not factorise)
Notice at the start, $t=0$, the accel $=72 \mathrm{~cm} / \mathrm{s} / \mathrm{s}$

$$
\begin{aligned}
& \text { At } t=3 \text {, the accel }=12 \times 3^{2}-72 \times 3+72=-36 \\
& \text { At } t=6 \text {, the accel }=12 \times 6^{2}-72 \times 6+72=-72
\end{aligned}
$$

To find at what times the accel will be zero, we solve:

$$
\begin{aligned}
& 12 t^{2}-72 t+72=0 \\
& t=\frac{72 \pm \sqrt{ }\left(72^{2}-4 \times 12 \times 72\right)}{2 \times 12} \\
& t=1.27 \text { sec and } 4.73 \mathrm{sec}
\end{aligned}
$$

We should compare graphs of $\boldsymbol{x}, \boldsymbol{v}$ and $\boldsymbol{a}$.


Max velocity is at $t=1.27 \mathrm{sec}$ and equals $41.6 \mathrm{~cm} / \mathrm{s}$
Min velocity is at $t=4.73 \mathrm{sec}$ and equals $-41.6 \mathrm{~cm} / \mathrm{s}$

Relating this to the graph, we can say that the car starts from rest, accelerates for 1.27 sec to a max speed of $41.6 \mathrm{~cm} / \mathrm{s}$ then starts to slow down and comes to rest momentarily at $t=3$ sec.
The car now goes backwards and reaches a speed of $41.6 \mathrm{~cm} / \mathrm{s}$ at $t=4.73$ seconds (actually the velocity is $\mathbf{- 4 1 . 6 \mathrm { cm } / \mathrm { s } \text { ) } { } ^ { ( 1 ) } ) ^ { 2 } )}$
The car then slows down, finally coming to rest at $O$ at $t=6 \mathrm{sec}$.
SPECIAL NOTE: Students find it very difficult to understand that when accel $=0$ the velocity is not zero too. Also, they find it hard to understand that the maximum speed is when the acceleration is zero!
This graph could help:

9. An ice cube with sides of 10 mm is melting at such a rate that the length, $\boldsymbol{x}$, of each side reduces by 1 mm every minute.
So $\boldsymbol{x}=\mathbf{1 0}-\boldsymbol{t}$
(a) Find the rate at which the AREA of a face is reducing.

$$
\begin{aligned}
A=x^{2} & =(10-t)^{2} \\
& =100-20 t+t^{2}
\end{aligned}
$$

Rate of decrease of area is $\underline{d A}=-20+2 t$

$$
d t
$$

Note: at $t=0 \quad \frac{d A}{d t}=-20 \mathrm{~mm}^{2} / \mathrm{min}$
but at $t=1 \min \frac{d A}{d t}=-18 \mathrm{~mm}^{2} / \mathbf{m i n}$
(b) Find the rate at which the VOLUME of the cube is reducing.

$$
\begin{aligned}
V=x^{3} & =(10-t)^{3} \\
& =(10-t)\left(100-20 t+t^{2}\right) \\
& =1000-300 t+30 t^{2}-t^{3}
\end{aligned}
$$

Rate of decrease of volume $=\frac{d V}{d t}=-300+60 t-3 t^{2}$
Note: at $t=0 \quad \frac{d V}{d t}=-300 \mathrm{~mm}^{3} / \mathrm{min}$
But at $t=1 \min \frac{d V}{d t}=-300+60-3=-243 \mathrm{~mm}^{3} / \mathrm{min}$
10. A simpler version of Qu 8 with "nice" numbers throughout is:

A model car moves in a straight line and its distance from the starting point is given by : $\boldsymbol{x}=\boldsymbol{t}(\boldsymbol{t}-\boldsymbol{6})^{2}$ The motion lasts from $\mathrm{t}=0$ to $\mathrm{t}=6 \mathrm{sec}$
(The graph would be a simple cubic curve.

The velocity will be a simple parabola

$$
\text { If } \begin{aligned}
x & =t(t-6)^{2} \\
& =t\left(t^{2}-12 t+36\right) \\
& =t^{3}-12 t^{2}+36 t \\
v & =3 t^{2}-24 t+36
\end{aligned}
$$

and the acceleration will be a line graph. accel, $\boldsymbol{a}=\boldsymbol{\sigma} \boldsymbol{t}-\mathbf{2 4}$
The velocity will be zero when $3 t^{2}-24 t+36=0$

$$
\begin{gathered}
\text { ie } 3\left(t^{2}-8 t+12\right)=0 \\
\text { ie } 3(t-2)(t-6)=0 \\
\text { so } t=2 \text { and } 6
\end{gathered}
$$

## Max distance from $O$ will be when $t=2$

Max dist $=2(2-6)^{2}=32$ metres.


The accel equals Zero when $\boldsymbol{t}=\boldsymbol{4}$ ( this is the point of inflection on $1^{\text {st }}$ graph at P )
This car start from O with an initial velocity of $36 \mathrm{~m} / \mathrm{s}$ and is slowing down until the speed becomes momentarily zero at $\mathrm{t}=2 \mathrm{sec}$, a maximum distance of 32 m from O . The car then goes backwards reaching a max speed of $12 \mathrm{~m} / \mathrm{s}$ at $\boldsymbol{t}=4 \mathrm{sec}$ (actually, a min Velocity of -12 )
It then slows down reaching O , stopping at $\mathrm{t}=6$.

## 11. An instructive teaching example.

A "remote controlled car" sets off from the base of a wall at W and accelerates up to a certain maximum speed. It then decelerates until it momentarily comes to rest and then accelerates in reverse along the same path, finally crashing into the wall at its original starting point W .
The equation for $S$, the distance from the wall in metres, at t seconds is :

$$
\begin{aligned}
& S=t^{2}(6-t) \text { for } 0 \leq t \leq 6 \\
& S=6 t^{2}-t^{3}
\end{aligned}
$$

Velocity $v=\frac{d S}{d t}=12 t-3 t^{2}$

$$
v=3 t(4-t)
$$

The velocity is zero at $t=0$ and 4 sec This means the car starts off from rest at $\mathrm{t}=0$ and at $\mathrm{t}=4$, it is at its maximum distance from the wall.
Max distance is $S=\mathbf{4}^{2}(6-4)$

$$
=32 \mathrm{~m}
$$

Acceleration $a=\frac{d v}{d t}=12-6 t$

Initially, although it starts from rest, it is accelerating at $12 \mathrm{~m} / \mathrm{s} / \mathrm{s}($ at $\mathrm{t}=0)$
(At $\mathrm{t}=1$ the accel is $6 \mathrm{~m} / \mathrm{s} / \mathrm{s}$ )

(6, -36)

The acceleration $\boldsymbol{a}=\mathbf{0}$ at $\boldsymbol{t}=\mathbf{2} \mathrm{sec}$
(The car then starts to decelerate and comes to instantaneous rest at $\mathrm{t}=4$ )

The point at which it stops accelerating and starts decelerating is called the point of INFLECTION.
ie $\boldsymbol{a}=\boldsymbol{0}$ at inflection point.
This is when the velocity is at its
Maximum. $\mathrm{v}_{\max }=3 \times 2(4-2)=12 \mathrm{~m} / \mathrm{s}$

The car hits the wall at $t=6 \mathrm{sec}$ with a velocity of $3 \times 6(4-6)$ $=-36 \mathrm{~m} / \mathrm{s}$

12. Students could test their understanding by answering the following:

A "remote controlled car" sets off from the base of a wall at W and accelerates up to a certain maximum speed. It then decelerates until it momentarily comes to test and then accelerates in reverse along the same path, finally crashing into the wall at its original starting point W .
The equation for $S$, the distance from the wall in metres, at $t$ seconds is :

$$
S=t^{2}(9-t) \quad \text { for } \quad 0 \leq t \leq 9
$$

(a) Find the velocity equation.
(i) Find the initial velocity
(ii) Find the time at which $\boldsymbol{v}=0$
(iii) Find the car's maximum distance from the wall.
(b) Find the acceleration equation.

(i) Find the initial acceleration.
(ii) Find the time at which the acceleration is zero.
(iii) Find the car's maximum velocity.
(iv) How far from the wall is the car when it stops accelerating and starts decelerating. (ie when $\boldsymbol{a}=0$ )
(c) At what time does the car hit the wall?
(d) At what speed does the car hit the wall?
(e) Draw careful graphs of distance, velocity and acceleration of the car for $t=0$ to 9 sec .

